An Alternating Least Squares (ALS) based Blind Source Separation Algorithm for Operational Modal Analysis

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Nomenclature

$\mathbf{x}$	State vector
$\mathbf{A_d}$	State matrix
$\mathbf{y}$	Measurement vector
$\mathbf{f}$	Unknown force vector
$\boldsymbol{\upsilon}$	Measurement noise vector
λ_i	i^{th} pole of the system
T_s	Sampling Period
$\mathbf{\Phi}$	Mode shapes of the system
$\mathbf{R_{yy}}$	Correlation matrix of the measurements
τ	Time lag
$\mathbf{R}_{yy}^{a}$	Augmented correlation matrix
$\mathbf{L^a}$, $\mathbf{R^a}$	Left and Right factors of augmented correlation matrix, containing information about mode shape.
$\mathbf{D}$	Diagonal matrix such that $\mathbf{R}_{yy}^{a} = \mathbf{L}^{a}\mathbf{D}[\tau]\mathbf{R}^{a}$, contains information about system poles.

ABSTRACT

In a former paper (“Second Order Blind Identification (SOBI) and its relation to Stochastic Subspace Identification (SSI) algorithm”, 28th IMAC, 2010), the authors established the link between the popular SSI algorithm used in output-only modal analysis and the Second Order Blind Identification (SOBI) algorithm developed for blind source separation in the field of signal processing. It was concluded that the two algorithms, although seemingly very different, are actually jointly diagonalizing the same covariance matrix over a range of time-lags. This is explicit in SOBI and implicit in SSI. One main difference, however, is that SOBI focuses on estimating the (real) modal matrix as a joint diagonalizer, but without taking advantage of the specific structure of the covariance matrix formed by the Markov coefficients and by incorrectly assuming no-damping or very low damping. On the other hand, SSI specifically exploits the covariance matrix structure so as to estimate complex modes, but puts less emphasis on the “joint diagonalizing” property of the modal matrix. The aim of this communication is to introduce a new algorithm based on Alternating Least Squares (ALS) approach that combines advantages of both SOBI and SSI in order to return improved estimates of modal parameters. It is shown in this work that this algorithm is capable of identifying complex modes, closely spaced modes and heavily damped and can also be expanded

to deal with the cases where there are less number of sensors available than the number modes to be estimated. The suggested approach therefore is a step towards expanding the applicability of BSS based approaches to Operational Modal Analysis applications.

1. Introduction

Several recent works have shown how second order blind source separation (SO-BSS) techniques, such as Second Order Blind Identification (SOBI) [1, 2], can be utilized for the purpose of output-only modal analysis [3-7]. However, in spite of encouraging results shown by these algorithms, applicability of BSS based algorithms for OMA purposes has been quite limited. This can be attributed to certain limitations associated with BSS based algorithms for OMA. Mathematical formulation of SO-BSS based OMA algorithms suggest that they are more suitable for lightly damped systems having real normal modal vectors. The fact that SO-BSS algorithms only estimate real modal vectors is a serious issue as this is seldom a case in real life. A methodology based on Hilbert transform is suggested in [8] to estimate complex mode shapes. However, the robustness of the method is yet to be ascertained. Yet another issue which restricts applicability of these algorithms is that they can only estimate as many modes as the number of output responses being measured.

Despite these limitations SO-BSS techniques present an interesting outlook with regards to operational modal analysis, as they differ from traditional OMA algorithms in terms of estimating the modal parameters of a system. In [9], the authors showed how SO-BSS algorithms, such as SOBI, are related to well known Stochastic Subspace Identification (SSI) algorithm [10, 11]. In this work it was shown that whereas SSI estimate the modes of a system by putting a constraint on the poles of the system, SO-BSS based algorithms use a joint diagonalization procedure to obtain the modal parameters by estimating orthogonal vectors that diagonalize the correlation matrices of observed responses. These orthogonal vectors are estimates of the modal vectors which are then used to obtain modal frequencies and damping by means of modal expansion theorem [12]. Since the two algorithms, SSI and SO-BSS, that share similar mathematical foundations, estimate modal parameters using different approaches, it is intriguing to pursue an algorithm that can combine the advantages of both the algorithms.

In this paper, an Alternative Least Squares (ALS) [13] based algorithm is proposed that combines the advantages of SO-BSS and SSI algorithms in order to overcome the limitations SO-BSS algorithms suffer from in terms of their application for OMA purposes. This algorithm can be explained within the framework of Parallel Factor (PARAFAC) theory [14]. Mathematical development of this algorithm is presented in the next section and preliminary results of this algorithm are shown in Section 3 by means of studies conducted on an analytical system. Finally, conclusions are made, in light of the results obtained and performance of the algorithm in general, with regards to its further development and suitability for OMA applications.

2. A PARAFAC based BSS Algorithm for OMA

2.1 General Background

This section recalls the main results and notations used in [8] as these results serve as the background and main motivation for the ALS based PARAFAC algorithm suggested in this paper.

Consider the following n degree of freedom (DOF) discrete-time state-space system,

$$\begin{cases} \mathbf{x}[k+1] = \mathbf{A}_d\mathbf{x}[k] + \mathbf{B}_d\mathbf{f}[k] \\ \mathbf{y}[k] = \mathbf{C}_d\mathbf{x}[k] + \mathbf{D}_d\mathbf{f}[k] + \mathbf{v}[k] \end{cases} \tag{1}$$

where $\mathbf{x}[k]$ is the $2n \times 1$ state vector, $\mathbf{y}[k]$ the $m \times 1$ measurement vector, $\mathbf{f}[k]$ the unknown force vector, $\mathbf{v}[k]$ the measurement noise vector, and $\mathbf{A}_d$ the $2n \times 2n$ state matrix. Let

$$\mathbf{A}_d = \mathbf{\Psi}\Sigma_d\mathbf{\Psi}^{-1} \tag{2}$$

be the eigenvalue decomposition of the state-matrix where the $2n \times 2n$ diagonal matrix, where

$$\Sigma_d = \begin{bmatrix} \exp\{\mathbf{\Lambda}T_s\} & \mathbf{0}_{n\times n} \\ \mathbf{0}_{n\times n} & \exp\{\mathbf{\Lambda}^*T_s\} \end{bmatrix} \tag{3}$$

(T_s = sampling period and *= conjugate operator) contains the modal parameters of interest, $\mathbf{\Lambda} = diag(\lambda_1,...,\lambda_n)$ with λ_i the i-th pole of the system, and

$$\mathbf{\Psi} = \begin{bmatrix} \mathbf{\Phi\Lambda} & \mathbf{\Phi}^*\mathbf{\Lambda}^* \\ \mathbf{\Phi} & \mathbf{\Phi}^* \end{bmatrix} \tag{4}$$

contains the mode shapes of the structure in the columns of $\mathbf{\Phi}$.

It has been shown in [8] that the correlation matrix of the measurements $\mathbf{y}[k]$ takes the following simple form under the assumption of white noise excitation.

$$\mathbf{R}_{yy}[\tau] = \mathbf{L}\Sigma_d^\tau \mathbf{R}, \quad \tau > 0 \tag{5}$$

where the left and right factors read

$$\mathbf{L} = \mathbf{C}_d \mathbf{\Psi} \in C^{m\times 2n}, \qquad \mathbf{R} = \mathbf{R}_{qq}[0]\mathbf{L}^t \in C^{2n\times m} \tag{6}$$

respectively, with $\mathbf{R}_{qq}[0]$ the (unknown) correlation matrix of the modal coordinate $\mathbf{q}[k] = \mathbf{\Psi}^{-1}\mathbf{x}[k]$. Such factorization of the correlation matrix, if it can be performed, returns all information about global parameters (modal frequency and damping) in the diagonal entries of Σ_d^τ, and the information about (unscaled) mode shapes through the left factor $\mathbf{L}$.

Since only m modes are recoverable in theory, one obvious shortcoming of the above approach is in cases where the number of measurements is less than twice the number of dof's of the system, i.e. $m < 2n$. This can however be dealt by considering the augmented correlation matrix as shown below

$$\mathbf{R}_{yy}^a[\tau] = \begin{bmatrix} \mathbf{R}_{yy}[\tau] & \cdots & \mathbf{R}_{yy}[\tau + K_2] \\ \vdots & & \vdots \\ \mathbf{R}_{yy}[\tau + K_1] & \cdots & \mathbf{R}_{yy}[\tau + K_1 + K_2] \end{bmatrix} \in IR^{mK_1 \times mK_2} \tag{7}$$

This augmented correlation matrix can be factorized as

$$\mathbf{R}_{yy}^a[\tau] = \begin{bmatrix} \mathbf{L} \\ \mathbf{L}\Sigma_d \\ \vdots \\ \mathbf{L}\Sigma_d^{K_1} \end{bmatrix} \Sigma_d^\tau \begin{bmatrix} \mathbf{R} & \Sigma_d \mathbf{R} & \cdots & \Sigma_d^{K_2}\mathbf{R} \end{bmatrix} = \mathbf{L}^a \Sigma_d^\tau \mathbf{R}^a \tag{8}$$

where $\mathbf{L}^a$ and $\mathbf{R}^a$ are now $mK_1 \times 2n$ and $2n \times mK_2$ matrices which contain all information about the m modes of the system provided that $mK_1 \geq 2n$ and $mK_2 \geq 2n$, with K_1 and K_2 set arbitrarily large by the user.

The next subsection describes an Alternating Least Squares (ALS) based PARAFAC algorithm that can achieve the proposed factorization on the empirical correlation matrix (or the augmented correlation matrix (Eqn. 7)) estimated from the measured data.

2.2 PARAFAC factorization

Let $\hat{\mathbf{R}}_{yy}^a[\tau]$ be the empirical correlation matrix estimated from the (finite-length) measurement vector $\mathbf{y}[k]$. The objective is then to find three matrices $\mathbf{L}^a$, $\mathbf{R}^a$ and $\mathbf{D}[\tau] = \Sigma_d^\tau$ such that the product $\mathbf{L}^a\mathbf{D}[\tau]\mathbf{R}^a$ is as close as possible to $\hat{\mathbf{R}}_{yy}^a[\tau]$ for a set of time-lags $\tau \in T = \{\tau_0, ..., \tau_K\}, \tau_K > ... > \tau_0 > 0$. Note that matrix $\mathbf{D}[\tau]$ is diagonal, but not semi-positive definite in general because its diagonal elements will contain an imaginary part as soon as damping is present in the system. For the same reason, the right factor $\mathbf{R}^a$ is in general different from the transpose of the left factor $\mathbf{L}^a$ and allowed to be complex-valued. This precludes the factorization to be solved by a joint diagonalization procedure, as done in SO-BSS and revisited in Ref. [9]. Clearly, this is a difficult non-linear problem, yet hopefully with a unique solution, provided that the set of considered time-lags T is large enough.

This type of problem has been studied in the statistical literature as parallel factor analysis or PARAFAC. In short, PARAFAC generalizes the eigenvalue decomposition of a matrix to a similar decomposition of a cube. The three dimensions of concern in operational modal analysis are the two spatial directions of the correlation matrix $\hat{\mathbf{R}}_{yy}^a[\tau]$ (column and raw indices corresponding to sensor labels) and the time-lag direction.

The investigation of efficient algorithms to solve the PARAFAC problem is a current and active topic of research [15][16]. For the sake of simplicity, the simplest algorithm is described here, based on alternative least squares (ALS) [13]. ALS consists in iteratively estimating one of the three matrices entering in the factorization assuming that the other two are fixed. This way, the minimization problem turns out linear in the parameters and is easily solved by least-squares. The algorithm is described in a step-by-step manner as follows:

1) Step $I = 0$: initialize $\mathbf{L}^a_{(i)}$, $\mathbf{R}^a_{(i)}$ and $\mathbf{D}[\tau]_{(i)}$

2) Step i+1 : update $\mathbf{L}^a_{(i)}$, $\mathbf{R}^a_{(i)}$ and $\mathbf{D}[\tau]_{(i)}$ as

a) $$\mathbf{L}^a_{(i+1)} = \arg\min_{\mathbf{L}^a} \sum_{\tau\in T} \left\| \hat{\mathbf{R}}^a_{yy}[\tau] - \mathbf{L}^a \mathbf{D}[\tau]_{(i)} \mathbf{R}^a_{(i)} \right\|^2 = \sum_{\tau\in T} \hat{\mathbf{R}}^a_{yy}[\tau] \mathbf{R}^{aH}_{(i)} \mathbf{D}[\tau]^H \left(\sum_{\tau\in T} \mathbf{D}[\tau] \mathbf{R}^a_{(i)} \mathbf{R}^{aH}_{(i)} \mathbf{D}[\tau]^H \right)^{-1} \tag{9}$$

b) $$\mathbf{R}^a_{(i+1)} = \arg\min_{\mathbf{R}^a} \sum_{\tau\in T} \left\| \hat{\mathbf{R}}^a_{yy}[\tau] - \mathbf{L}^a_{(i)} \mathbf{D}[\tau]_{(i)} \mathbf{R}^a \right\|^2 = \left(\sum_{\tau\in T} \mathbf{D}[\tau]^H \mathbf{L}^{aH}_{(i)} \mathbf{L}^a_{(i)} \mathbf{D}[\tau] \right)^{-1} \sum_{\tau\in T} \mathbf{D}[\tau]^H \mathbf{L}^{aH}_{(i)} \hat{\mathbf{R}}^a_{yy}[\tau] \tag{10}$$

c) $$\mathbf{D}^a_{(i+1)} = \arg\min_{\mathbf{D}^a} \sum_{j=1}^{mK_2} \left\| \tilde{\mathbf{R}}^a_{yy}[j] - \mathbf{D}\tilde{\mathbf{R}}[j] \mathbf{L}^{a\,t}_{(i)} \right\|^2 = \sum_{j=1}^{mK_2} \tilde{\mathbf{R}}^a_{yy}[j] \mathbf{L}^{a\,*}_{(i)} \tilde{\mathbf{R}}[j]^H \left(\sum_{j=1}^{mK_2} \tilde{\mathbf{R}}[j] \mathbf{L}^{a\,t}_{(i)} \mathbf{L}^{a\,*}_{(i)} \tilde{\mathbf{R}}[j]^H \right)^{-1} \tag{11}$$

where, in the last line,

$$\tilde{\mathbf{R}}^a_{yy}[j] = \begin{bmatrix} \mathbf{R}^a_{yy}[\tau_0]_{1,j} & \cdots & \mathbf{R}^a_{yy}[\tau_0]_{mK_1,j} \\ \vdots & & \vdots \\ \mathbf{R}^a_{yy}[\tau_K]_{1,j} & \cdots & \mathbf{R}^a_{yy}[\tau_K]_{mK_1,j} \end{bmatrix} \in IR^{K\times mK_1},$$

$$\tilde{\mathbf{R}}[j] = \begin{bmatrix} \mathbf{R}_{1,j} & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \mathbf{R}_{2n,j} \end{bmatrix} \in C^{2n\times 2n},\ j = 1,...,mK_2, \quad \text{and} \quad \mathbf{D}^a = \begin{bmatrix} \Sigma_1^{\tau_0} & \cdots & \Sigma_{2n}^{\tau_0} \\ \vdots & & \vdots \\ \Sigma_1^{\tau_K} & \cdots & \Sigma_{2n}^{\tau_K} \end{bmatrix} \in C^{K\times 2n_1}$$

3) Stop iterating if the relative errors $\left\| \mathbf{L}^a_{(i=1)} - \mathbf{L}^a_{(i)} \right\|$, $\left\| \mathbf{R}^a_{(i+1)} - \mathbf{R}^a_{(i)} \right\|$ and $\left\| \mathbf{D}^a_{(i+1)} - \mathbf{D}^a_{(i)} \right\|$ are all smaller than a predefined value.

At this point, it is important to mention a few remarks about the convergence of the ALS algorithm, which also happens to be one the limitations of the suggested approach.

- First it is clear that there is no guarantee of convergence to a global minimum. Hence, the final estimates will strongly depend on the quality of the starting values. It has been observed by the authors that reasonable estimates are returned after initializing matrix **D** with the resonance frequencies obtained from a simple peak-picking method, with assumed zero damping ratios. Matrices $\mathbf{L}^a$ and $\mathbf{R}^a$ have been initialized randomly, although better strategies could perhaps be investigated such as using results from SO-BSS.

- Second, ALS is known to exhibit a low convergence speed, which is the price to pay for its simplicity. Accelerating strategies have recently been proposed in the literature, but this is outside the scope of this paper. Indeed, it is underlined that ALS is only one possible technique to solve the PARAFAC problem, and that more efficient algorithms are expected to come in the near future. Hence, this should not lower the interest of the proposed approach.

One apparent weakness of the proposed PARAFAC approach, however, is that it places no constraint on the estimated **D** matrix to be an actual correlation matrix, i.e. with elements in the form $\exp\{\lambda_i \tau T_s\}$ or conjugate of. But this is also where the limit lays between a fully BSS approach and an *ad hoc* approach such as SSI purposely designed for operational modal analysis. Based on this work, the authors are of the opinion that involving further efforts into forcing such a constraint would quickly drive the analysis into the realm of SSI algorithms, where the simplicity of BSS approaches would be lost.

3. Results and Discussions

A simple 3 degree of freedom (DOF) system with following [M], [C] and [K] matrices is used in this study. It should be noted that the damping matrix [C] is chosen randomly.

$$M=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} \qquad K=\begin{bmatrix}100 & -50 & 0\\ -50 & 100 & -50\\ 0 & -50 & 50\end{bmatrix} \qquad C=\begin{bmatrix}1.1334 & 0.2764 & 0.1217\\ 0.1846 & 1.2215 & 0.2806\\ 0.2376 & 0.0529 & 1.2751\end{bmatrix}$$

Table 1 shows the theoretical eigen frequencies and corresponding modal shapes of the system. Note that the mode shapes are normalized with respect to the first DOF.

Table 1: Theoretical System Parameters (Frequency, Damping and Mode Shapes)

Eigen Freq	Freq (Hz)	Damping (%)	DOF 1	DOF 2	DOF 3
-0.1244 + 0.4852i	0.4852	24.83	1 + 0i	1.7959 + 0.0205i	2.2325 + 0.0488i
-0.0861 + 1.4005i	1.4005	6.14	1 + 0i	0.4438 + 0.0133i	-0.8033 + 0.0096
-0.0784 + 2.0265i	2.0265	3.86	1 + 0i	-1.2454 - 0.0327i	0.5512 + 0.0521i

The system is excited by means of a random uncorrelated set of input at all 3 degrees of freedom. The response time history and corresponding auto spectra is shown in Figure 1.

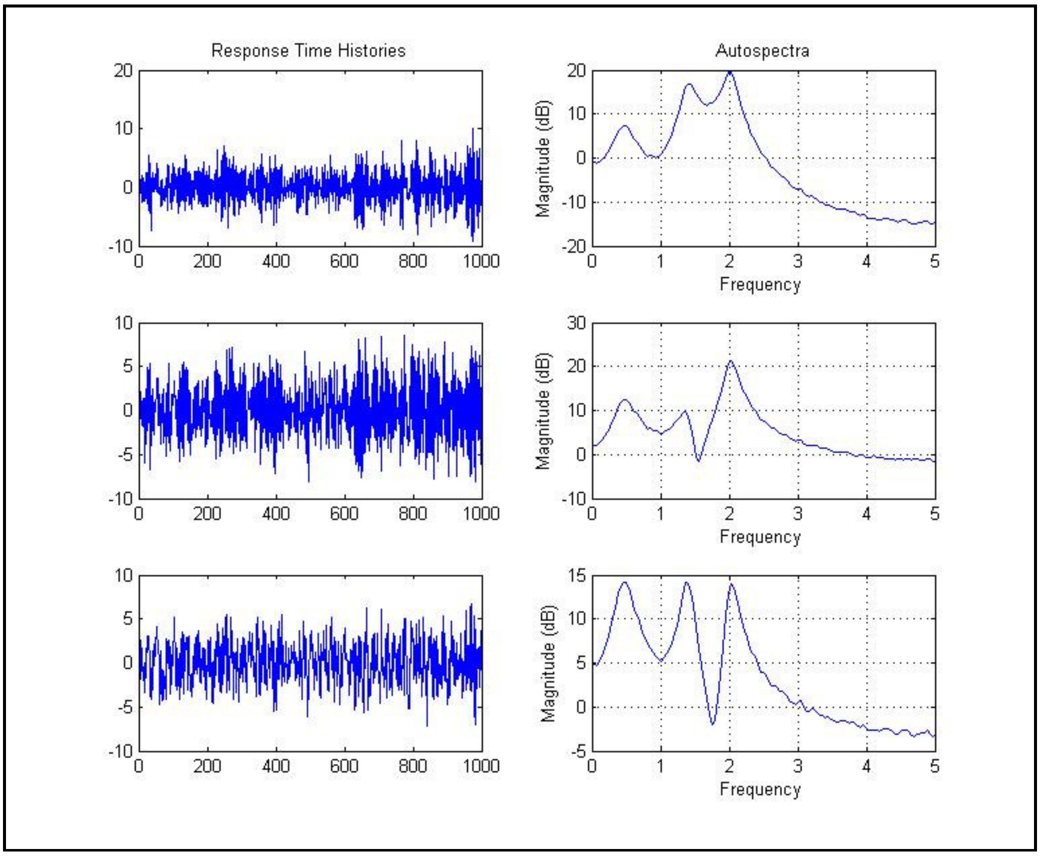

Figure 1: Response time histories and corresponding autospectra

As mentioned in Section 2.2, ALS based PARAFAC algorithm requires initial estimates of ***L***, ***D*** and ***R*** matrices. For this study ***L*** and ***R*** and initialized by means of random complex matrices and ***D*** is initialized by means of peak picking the frequencies from the summation of the three autospectra. It should be noted that regular joint diagonalization as in SOBI can be used as a preprocessing step to get the initial estimates of ***D***, however this step is not performed in present study.

Table 2 compares results obtained from ALS algorithm with theoretical values of modal parameters.

Table 2: Comparison of Estimated (A) and Theoretical (T) Modal Parameters (Frequency, Damping, Mode Shapes)

Mode #		Freq (Hz)	Damping (%)	DOF 1	DOF 2	DOF 3
Mode 1	T	0.4852	24.83	1 + 0i	1.7959 + 0.0205i	2.2325 + 0.0488i
	A	0.4883	25.50	1 + 0i	1.7955 + 0.0226i	2.2231 + 0.0499i
Mode 2	T	1.4005	6.14	1 + 0i	0.4438 + 0.0133i	-0.8033 + 0.0096i
	A	1.4060	6.28	1 + 0i	0.4443 + 0.0257i	-0.8041 + 0.0192i
Mode 3	T	2.0265	3.86	1 + 0i	-1.2454 - 0.0327i	0.5512 + 0.0521i
	A	2.0310	4.01	1 + 0i	-1.2392 - 0.0312i	0.5505 + 0.0414i

Mode shape comparison is also shown by means of Figure 2, which shows very good agreement with theoretical mode shapes.

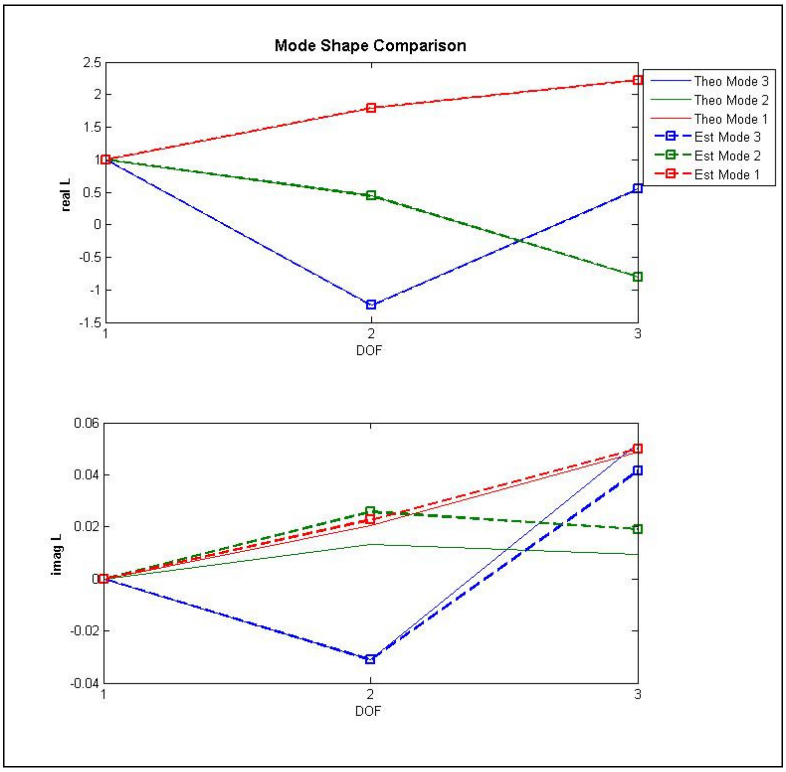

Figure 2: Mode Shape Comparison

The performance of ALS algorithm with regards to estimation of modal frequency and damping can also be evaluated by comparing the estimated ***D*** matrix with theoretical solution. This is shown in Figure 3.

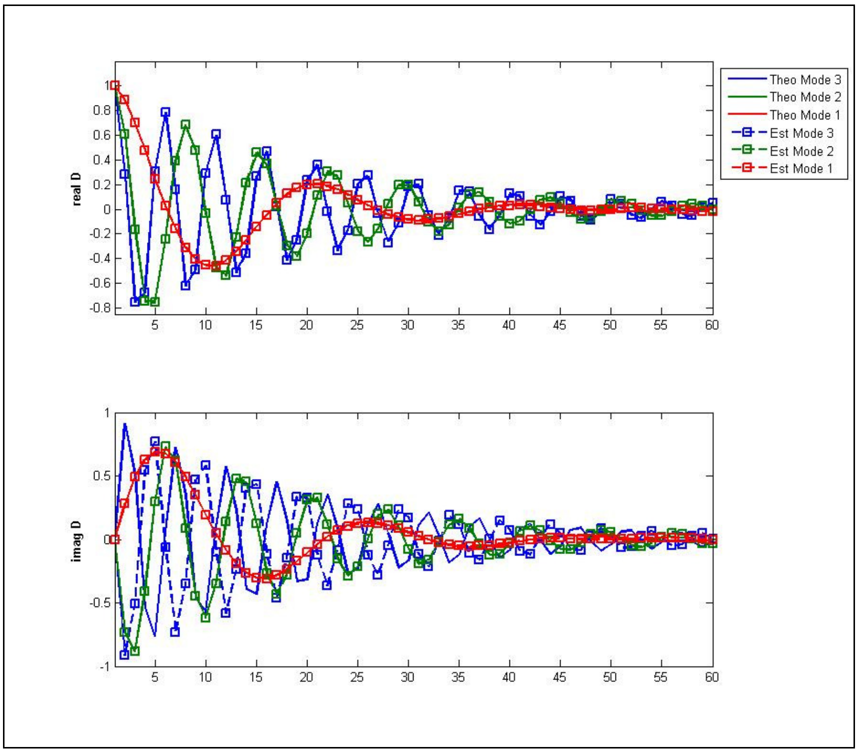

Figure 3: Comparison of D matrices

Modal parameters estimated using the proposed algorithm are in very good agreement with the theoretical modal parameters. These simulations show that the algorithm handles the heavily damped modes very well and is also able to estimate complex mode shapes. This is a definite improvement over SO-BSS algorithms such as SOBI whose performance is not satisfactory while dealing with cases of heavy damping or complex mode shapes. Based on these results it can be said that the performance of the suggested algorithm is very encouraging and augurs well for further development of the algorithm.
Further development of this algorithm needs to address the limitations which this algorithm suffers in its current formulation. These limitations are on account of the simple ALS based approach used for factorizing the covariance matrices using PARAFAC framework. First major concern is convergence of this algorithm. The fact that since the ALS algorithm requires initial estimates of ***L***, ***D*** and ***R***, it is possible that if these initial estimates are not chosen carefully, the algorithm might not converge or converge to a local minimum. To avoid this scenario, it is suggested that SOBI can be performed as a pre-processing step and its results can be used as initial estimates of ***L***, ***D*** and ***R***. In the current simulation, even when only ***D*** is initialized with intelligent estimates based on peak-picking the frequencies (***L*** and ***R*** are initialized randomly), the results are very satisfactory. However, it needs to be verified if this procedure can be generalized to work for all kind of systems and situations. The performance of the algorithm is also required to be evaluated in presence of noisy data and for some real world cases, to have a more definite word regarding its suitability for OMA. As mentioned before, ALS is a very simple algorithm for performing PARAFAC based factorization. A more robust algorithm with better convergence properties might be another step in this research.

4. Conclusions

This paper proposes a Parallel Factor based Alternating Least Squares Blind Source Separation algorithm for Operational Modal Analysis applications. Development of this algorithm follows from the previous work by the authors that establishes and explores the fundamental relationship between second order BSS algorithms and Stochastic Subspace Iteration algorithm. It is shown by means of a simulated system that this algorithm is capable of estimating heavily damped modes and complex mode shapes. These results are very encouraging and serve as great motivation to further improve and optimize the proposed algorithm and test it on more realistic scenarios in order to use it for OMA purposes.

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