Quantifying uncertainty in modal parameters estimated using higher order time domain algorithms

S. Chauhan

Bruel & Kjær Sound and Vibration Measurement A/S
Skodsborgvej 307, DK 2850, Naerum, Denmark

Email: schauhan@bksv.com

Nomenclature

$\{X(s)\}$, $\{F(s)\}$	Response and Force vector in Laplace domain
[M], [C] and [K]	Mass, Damping and Stiffness matrices
N	Degrees of freedom
[H(s)]	Frequency response function matrix in Laplace domain
$[\ ]^H$	Hermitian of a matrix
$[\alpha_k]$, $[\beta_k]$	Polynomial coefficient matrices
$[h(t)]$	Impulse response function matrix
$\widehat{\ }$	Estimate of a quantity
$[\ ]^+$	Pseudo-inverse of a matrix
$[\varepsilon]$	Residual (Noise or Error) matrix
$\Sigma_{\hat{\varepsilon}}$	Covariance matrix of noise or residuals
$\Sigma_{\hat{A}}$	Covariance Matrix of polynomial coefficient matrices
$\otimes$	Kronecker product
vec( )	Vectorization operator
t	t-ratio
σ	Standard deviation

ABSTRACT

Modal parameters (natural frequency, damping and mode shapes) play an important role in dynamic characterization of a structure. These parameters are estimated using advance parameter estimation algorithms. However, the estimated modal parameters are often quoted without much statistical evaluation of the estimation procedure. Since, modal parameters are estimated from measured data and the estimation procedure itself is often an error minimization procedure (like Least Squares approach), it is necessary to quantify the uncertainty associated with the parameter estimation procedure. One way to achieve this goal is by providing confidence intervals for the estimated modal parameters.

This paper addresses the issue of uncertainty quantification for modal parameters estimated using high order time domain algorithms. A methodology for estimating confidence intervals for the estimated modal parameters is presented and its usage is illustrated by means of simulated and experimental examples.

Keywords: Uncertainty quantification, confidence intervals, modal parameter estimation, Polyreference time domain, error estimation

1. Introduction

Goal of modal parameter estimation is to estimate modal parameters (natural frequency, damping and mode shape) of the system/structure being analyzed, given the measured data (input-output or output only). There are several algorithms available to carry out this task. The procedure of estimating modal parameters from measured data typically involves two stages; i) Preprocessing raw data, using signal processing techniques, into characteristic functions (for e.g. Frequency Response Functions or Impulse Response Functions for experimental modal analysis or covariances for operational modal analysis), and ii) Application of modal parameter estimation techniques to characteristic functions. The result of second stage is estimation of modal parameters characterizing the structure. The above mentioned two-stage procedure for estimating modal parameters can also be viewed within a statistical framework as most algorithms and techniques are based on solid statistical foundation. This applies to both signal processing techniques (for e.g. averaging techniques for reducing random errors [1]) and parameter estimation algorithms (for e.g. Least Squares approach is a key step in most algorithms). In spite of this, the estimates are often provided as it is without any statistical insight about the estimates or the estimation procedure.

The statistical nature of modal parameter estimation procedure underlines the need for providing statistical evaluation of the procedure and the estimated quantities (modal parameters). This is important because raw data, which forms the starting point of the estimation procedure, has often associated with it the issues of accuracy and sufficiency. These issues, related to the raw data, affect the successive procedures and quantities that are estimated from it and hence the estimated quantities, such as modal parameters, have certain uncertainties associated with them. A true reflection, of how accurate and reliable the estimates are, can only be provided by including some measure of quantifying the associated uncertainty.

Uncertainty quantification in modal parameter estimation has not received much attention in the past though there are some works that have explored this area in recent times. In [2], covariance matrices of estimates obtained from Maximum Likelihood and prediction error based parameter estimation methods are used to obtain the uncertainty bounds. [3] uses a similar approach but within OMA framework to obtain uncertainty bounds for modal parameters estimated using Stochastic subspace identification (SSI) algorithm [4]. The methodology described in [3] was developed for OMA data acquired in a single experiment. In [5] the method was extended to multi-setup scenario.

One way of expressing the uncertainty associated with the estimated modal parameters is by means of confidence intervals [6]. This paper presents a theoretical framework for quantifying the uncertainty associated with modal parameters estimated using higher order time domain modal parameter estimation algorithms such as Polyreference Time Domain algorithm [7, 8].

The paper is organized in the following manner. Section 2 presents the theoretical framework for obtaining the confidence intervals for the estimated modal parameters. It is divided in two subsections; section 2.1 presents the modal parameter estimation procedure from a statistical perspective, looking at it purely from the point of view of a least squares problem, a well-known statistical procedure. It establishes mathematical formulas for calculating the covariance matrix of residuals (or noise) and that of polynomial coefficient matrices. These covariance matrices are then used in Section 2.2, which explains the methodology to obtain confidence intervals of the estimated modal parameters i.e. modal frequency, modal damping and mode shape. In Section 3 the proposed methodology is validated and demonstrated by means of simulated studies carried out on a 5 DOF (degrees of freedom) analytical system. Finally conclusions are provided in section 4.

2. Theoretical Background

2.1 Modal Parameter Estimation Procedure

The characteristic equation for a general mechanical vibration system having N degrees of freedom (DOF) is given by following expression.

$$[\mathrm{H(s)}] = \{X(\mathrm{s})\}/\{F(\mathrm{s})\} = 1/|[\mathrm{M}]\mathrm{s}^2 + [\mathrm{C}]\mathrm{s} + [\mathrm{K}]| \tag{1}$$

where [M], [C] and [K] are mass, damping and stiffness matrices of size NXN, $\{X\}$ and $\{F\}$ are N dimensional response and excitation force vectors and | | represents determinant of the quantity. [H] is an NxN matrix called *frequency response function* (FRF) matrix. The modal parameters (i.e. modal frequency, damping and mode shapes), representing the dynamic characteristics of the system, can be obtained from the 2N eigenvalues and eigenvectors of this equation.

Since the mass, stiffness and damping matrices are not available in practice, typically measured data is utilized to first estimate FRFs and then modal parameter estimation algorithms are applied on estimated FRFs to estimate the modal parameters. H_1 estimator [1] (represented by following formulation in Laplace domain) is a commonly used method for estimating FRFs from the measured data. Note that other signal processing techniques such as averaging and windowing are also used simultaneously.

$$[\mathrm{H(s)}] = [X(\mathrm{s})F(\mathrm{s})^{\mathrm{H}}][F(\mathrm{s})F(\mathrm{s})^{\mathrm{H}}]^{-1} \tag{2}$$

Above Equation can be written in matrix coefficient model form in the following manner [1-3]

$$[\mathrm{H(s)}] = \frac{\sum_{k=0}^{m-1} s^k[\beta_k]}{\sum_{k=0}^{m} s^k[\alpha_k]} \tag{3}$$

In the above equation, m is referred to as the modal order of the System. The total number of modes (or roots) of the system is related to the modal order and the size of the coefficient matrices α. It turns out that, theoretically, product of modal order and size of coefficient matrices α is equal to total number of modes, i.e. 2N. Please refer [2,3] for more details.

Eq. 3) can also be represented in time domain by means of Impulse response functions (using inverse Fourier transformation).

$$\sum_{k=0}^{m} [\alpha_k]\,[h(t_k)] = 0 \tag{4}$$

The impulse response matrix $[h]$ is also an NxN matrix though coefficient matrices α in case of Eq. 3) and Eq. 4) are different.

Eq. 3) and 4) form the basis of most parameter estimation algorithms. The task is to first estimate the coefficient matrices and then obtain the roots, or the modal parameters of the system by using Eigenvalue decomposition of companion matrix formed from the estimated coefficient matrices [2-4].

As mentioned before, the goal of this paper is uncertainty quantification by means of establishing confidence intervals of the estimated modal parameters. High order time domain algorithms such as Polyreference Time Domain (PTD) algorithm [7, 8], are well known in the industry and are used extensively for modal parameter estimation. Thus, in this paper estimation of confidence intervals is demonstrated using PTD as the basis for modal parameter estimation.

As a starting point for PTD, Eq. 4) can be expanded, for a particular order m, as follows:

$$[h(t_{i+0})][\alpha_0] + [h(t_{i+1})][\alpha_1] + [h(t_{i+2})][\alpha_2] + ------ + [h(t_{i+m})][\alpha_m] = [0] \tag{5}$$

where $[h(t_k)]$ is a N_o x N_i impulse response matrix at time instant t_k and $[\alpha_k]$ is the N_i x N_i polynomial coefficient matrix. Above equation can be remodeled by normalizing it with $[\alpha_m]$ and writing it as

$$[[h(t_{i+0})] \quad [h(t_{i+1})] \quad \cdots \quad [h(t_{i+m-1})]]_{\mathrm{N_o}\times\mathrm{N_i m}} \begin{bmatrix} [\alpha_0] \\ [\alpha_1] \\ \vdots \\ [\alpha_{m-1}] \end{bmatrix}_{\mathrm{N_i m}\times\mathrm{N_i}} = -[h(t_{i+m})]_{\mathrm{N_o}\times\mathrm{N_i}} \tag{6}$$

or in a more compact way as

$$[p]_i\,[A]=[q]_{\mathrm{i}} \tag{7)}$$

Since polynomial coefficient matrices [α_k] are constant for Linear Time Invariant system, several equations similar to Eq. 7) can be formed by changing the value of i and then the equations are solved in a Least Square (LS) [9, 10] manner to obtain the estimate of polynomial coefficients or matrix [A].

$$[P][A]=[Q] \tag{8)}$$

At this point it is important to note why the LS solution is required. It is well known that measured data is always contaminated with noise and irrespective of signal processing techniques used, functions (such as IRFs) estimated from the measured data are also subject to error due to noise present in the measured data. In other words, there is some uncertainty in the IRFs, which in turn creeps in to the modal parameter estimation procedure and also affects the modal parameters estimated from them. Least squares approach to solve Eq. 8, is one of the procedures to estimate the polynomial coefficient matrices such that the error is minimized. Hence, to quantify the uncertainty, which is representative of the error, it is critical to represent Eq. 8) by including the error term as shown below.

$$[P][A]=[Q]+[\varepsilon] \tag{9)}$$

An estimate of polynomial coefficient matrices using the LS estimation is given by the following equation, where is [$\hat{A}$] used to denote the estimate of [A], [Q_m] is defined as $[Q_m]=[Q]+[\varepsilon]$ and $^+$ represents the pseudoinverse.

$$\begin{aligned}[\hat{A}]&=\left(\left([P]^T[P]\right)^{-1}[P]^{\mathrm{T}}\right)[Q_m]\\ &\text{or}\\ [\hat{A}]&=[P]^{+}\,[Q_m]\end{aligned} \tag{10)}$$

From the model ([$\hat{A}$]) estimated in Eq. 10), the target data [Q_m] can be predicted. Thus, predicted data (represented by [$\hat{Q}$]) is a function of [$\hat{A}$] and [P].

$$[\hat{Q}]=f\left(P,\hat{A}\right)=[P][\hat{A}] \tag{11)}$$

The difference between target values (or actual values) [Q_m] and predicted values [$\hat{Q}$] are called ***Residuals*** and can be looked upon as an estimate of the error term $[\varepsilon]$, defined in Eq. 9), and can be represented as $[\hat{\varepsilon}]$.

$$[\hat{\varepsilon}]=[Q_m]-[\hat{Q}] \tag{12)}$$

For calculating confidence intervals following two quantities are required.

1. Covariance Matrix of Noise or Residuals ($\Sigma_{\hat{\varepsilon}}$)
2. Covariance Matrix of polynomial coefficient matrices ($\Sigma_{\hat{A}}$)

2.1.1 Covariance Matrix of Noise or Residuals ($\Sigma_{\hat{\varepsilon}}$)

To calculate $\Sigma_{\hat{\varepsilon}}$, first the following moment matrices are defined

$$\begin{aligned}[G_{PP}]&=[P]^T[P]\\ [G_{PQ}]&=[P]^T[Q_m]\\ [G_{QQ}]&=[Q_m]^T[Q_m]\end{aligned} \tag{13)}$$

It is easy to follow that based on above definitions, Eq. 10) can be expressed as

$$[\hat{A}]=[G_{PP}]^{-1}\,[G_{PQ}] \tag{14)}$$

The residual covariance matrix ($\Sigma_{\hat{\varepsilon}}$) can be calculated as

$$\Sigma_{\hat{\varepsilon}} = [\hat{\varepsilon}]^T [\hat{\varepsilon}] = \left([Q_m] - [\hat{Q}]\right)^{\mathrm{T}} \left([Q_m] - [\hat{Q}]\right) \tag{15}$$

By substituting [$\hat{Q}$] from Eq. 11) in Eq. 15) and simplifying using Eq. 13), $\Sigma_{\hat{\varepsilon}}$ is defined in terms of moment matrices as

$$\Sigma_{\hat{\varepsilon}} = \left([G_{QQ}] - [G_{PQ}]^T [G_{PP}]^{-1} [G_{PQ}]\right) \tag{16}$$

2.1.2 Covariance Matrix of Polynomial Coefficient Matrices ($\Sigma_{\hat{A}}$)

For calculating $\Sigma_{\hat{A}}$, Eq. 10) can be reframed using Eq. 9) and written as

$$[\hat{A}] - [A] = [G_{PP}]^{-1} [P]^{\mathrm{T}} [\varepsilon] \tag{17}$$

It is assumed that the error is distributed normally. This ensures the consistency and asymptotically normal distribution of polynomial coefficient estimates and that the estimation errors, i.e. difference between estimated polynomial coefficients ($[\hat{A}]$ or $\hat{\alpha}$) and true polynomial coefficients ($[A]$ or α), are distributed normally with zero mean and covariance matrix $\Sigma_{\hat{A}}$ [9, 11, 12].

$\Sigma_{\hat{A}}$ can be now calculated by using vector notation of $[\hat{A}] - [A]$ and calculating

$$\Sigma_{\hat{A}} = \left(vec\left([\hat{A}] - [A]\right)\right)^T \left(vec\left([\hat{A}] - [A]\right)\right)$$

which simplifies to [11]

$$\Sigma_{\hat{A}} = [G_{PP}]^{-1} \otimes [\Sigma_{\hat{\varepsilon}}] \tag{18}$$

2.2 Estimation of Confidence Intervals

In statistics, confidence interval of an estimated quantity $\hat{\theta}$ is calculated from the distribution of t-ratio using the following equation [13]

$$\hat{\theta}_{\pm} = t \times \hat{\sigma}_{\hat{\theta}} \tag{19}$$

where $\hat{\sigma}_{\hat{\theta}}$ is the estimate of ***standard deviation*** or the ***standard error*** of $\hat{\theta}$ and t is ***t-ratio*** (also called ***Confidence coefficient***) that depends on the distribution assumed and whose value can be obtained for the required ***confidence level*** (for e.g. 95% confidence level) from a standard table. Thus for estimating the confidence intervals of an estimated quantity one needs to know

1. The standard deviation or variance of the estimated quantity, and
2. The t-ratio, based on the assumed distribution, corresponding to the required confidence level.

The challenge in calculating confidence intervals for the estimated modal parameters, using the above described approach, comes from the fact that estimating standard deviation of estimated modal parameters is not a straightforward task. This is due to the fact that modal parameters are indirect result of parameter estimation procedure. They are calculated from the estimated polynomial coefficient matrices and are nonlinear functions of the same. Thus in order to calculate standard deviation associated with estimated modal parameter, one needs to first understand the propagation of uncertainty through the functional relationship between the polynomial coefficient matrices and the modal parameters.

If θ is a quantity that is continuous function of another quantity β and $\hat{\theta}$ is the estimate of θ evaluated at the estimate $\hat{\beta}$, then the covariance matrix of $\hat{\theta}$ ($\Sigma_{\hat{\theta}}$) in terms of covariance matrix $\Sigma_{\hat{\beta}}$ is given as [14]

$$\Sigma_{\hat{\theta}} = \left(\frac{\partial \theta}{\partial \beta}\right)_{\hat{\beta}}^{\mathrm{T}} \Sigma_{\hat{\beta}} \left(\frac{\partial \theta}{\partial \beta}\right)_{\hat{\beta}} \tag{20)}$$

Using this equation, one can estimate the covariance matrix (and subsequently standard deviation) associated with modal parameters from the knowledge of covariance matrix of polynomial coefficient matrices and their derivative functions with respect to polynomial coefficients. However, it is important to note that in case of modal parameters Eq. 20) is approximate and only holds up to higher terms, due to the nonlinear functional relationship.

To reiterate, in order to estimate the confidence intervals associated with the estimated modal parameters, one requires:

1. Covariance matrix of polynomial coefficients, i.e. $\Sigma_{\hat{A}}$ (which has already been derived in Eq. 18),
2. Derivative functions (Gradients) of modal parameters (modal frequency, damping and mode shapes) with respect to polynomial coefficients, evaluated at the estimated polynomial coefficients, and
3. The t-ratio: It is typically assumed that the t-ratio follows the Student's t distribution [11]. This is also the approach taken in this work. However, it is also common to use Chi-Square distribution [3].

Once these quantities are estimated, Eq. 19) can be utilized for calculating the confidence intervals of the estimated modal parameters.

3. Results

In this section, the theory described in Section 2 is validated by means of studies conducted on a 5 DOF analytical system. The analytical 5 DOF system used in this study is constructed using the following [M], [C] and [K] matrices.

$$[M] = \begin{bmatrix} 250 & 0 & 0 & 0 & 0 \\ 0 & 350 & 0 & 0 & 0 \\ 0 & 0 & 30 & 0 & 0 \\ 0 & 0 & 0 & 450 & 0 \\ 0 & 0 & 0 & 0 & 50 \end{bmatrix} \qquad [C] = \begin{bmatrix} 3250 & -250 & 0 & 0 & 0 \\ -250 & 450 & -200 & 0 & 0 \\ 0 & -200 & 320 & -120 & 0 \\ 0 & 0 & -120 & 190 & -70 \\ 0 & 0 & 0 & -70 & 270 \end{bmatrix}$$

$$[K] = \begin{bmatrix} 9000 & -5000 & 0 & 0 & 0 \\ -5000 & 11000 & -6000 & 0 & 0 \\ 0 & -6000 & 12500 & -6500 & 0 \\ 0 & 0 & -6500 & 14500 & -8000 \\ 0 & 0 & 0 & 0 & 15000 \end{bmatrix} \times 1000$$

Modal parameters of the system (modal frequencies, damping and mode shapes) are listed in Table 1 and 2.

Table 1: Modal frequencies and damping of the 5 DOF analytical system

Frequency (Hz)	**Damping (%)**
12.5263	1.1486
22.0830	1.0589
34.8635	2.1720
88.5238	0.4872
104.779	0.8473

Table 2: Mode Shapes of the 5 DOF analytical system

Freq/DOF	**12.52 Hz**	**22.08 Hz**	**34.86 Hz**	**88.52 Hz**	**104.77 Hz**
1	1 + 0i	1 + 0i	1 + 0i	1 + 0i	1 + 0i
2	1.489 + 0.038i	0.837 + 0.064i	-0.601 + 0.045i	-13.66 + 0.591i	-19.86 + 0.717i
3	1.360 + 0.046i	-0.238 + 0.029i	-0.249 + 0.013i	220.4 - 11.86i	464.6 - 19.56i
4	1.201 + 0.051i	-1.209 - 0.006i	0.132 - 0.016i	121.9 - 4.464i	-17.66 + 0.872i
5	0.654 + 0.027i	-0.689 - 0.004i	0.084 - 0.010i	-2079.6 + 68.45i	21.14 - 1.517i

The analytical system is analyzed using Monte Carlo simulations to validate the confidence intervals (of modal parameters) obtained using equations developed in Section 2. A total of 500 simulation runs are conducted. In each simulation, the system is excited at all DOFs by means of a different random force realization and response to this excitation force is collected at all DOFs. For each run, the Frequency Response Functions (FRFs) are calculated, from the simulated force and response time histories, using H_1 estimator and Impulse Response Functions (IRFs) are generated by inverse Fourier transformation of FRFs. It should be noted that same number of time samples are generated for each run. The sampling frequency is 256 Hz and signal processing parameters such as windowing, blocksize, overlap, etc. are kept same from one simulation run to another. Finally Polyreference Time Domain algorithm (PTD) is used to estimate the modal parameters from the IRFs. The algorithm is evaluated at order 8.

The confidence intervals (CIs) of modal parameters are estimated for each of the 500 runs along with the estimates. The CIs obtained for each run, for modal frequencies and damping, are first converted to standard deviation (See Eq. 19). To validate the results, the mean of these standard deviations of modal frequency and damping are compared to sample standard deviation of these quantities based on the 500 samples from the Monte Carlo simulation runs. The results are listed in Table 3 along with the mean estimates of modal frequency and damping for each of the five modes.

Table 3: Result of Monte Carlo Simulation Studies

	Mean Frequency (Hz)	$\boldsymbol{\sigma_f}$	$\boldsymbol{(\sigma_f)_{mean}}$	**Mean Damping (%)**	$\boldsymbol{\sigma_d}$	$\boldsymbol{(\sigma_d)_{mean}}$
Mode 1	12.5304	0.0091	0.0108	1.2024	0.0610	0.0768
Mode 2	22.0802	0.0090	0.0058	1.0902	0.0328	0.0263
Mode 3	34.8628	0.0031	0.0024	2.1798	0.0085	0.0069
Mode 4	88.5190	0.0084	0.0081	0.4966	0.0083	0.0092
Mode 5	104.775	0.0046	0.0053	0.8495	0.0060	0.0063

σ_f and σ_d: Sample standard deviation of estimated frequency and damping, $(\sigma_f)_{mean}$ and $(\sigma_d)_{mean}$: Mean of estimated standard deviation of frequency and damping at each simulation run.

It can be seen from the results listed in Table 3 that standard deviations of modal frequency, $(\sigma_f)_{mean}$ and modal damping, $(\sigma_d)_{mean}$ estimated using the theory discussed in section 2 compare very well with the sample standard deviations (σ_f and σ_d) for each mode.

Having validated the confidence interval estimation procedure, the focus is now shifted to observing the difference in confidence intervals of actual system modes and those of mathematical (or computational) modes. From statistical point of view, it is expected that confidence intervals of the actual system modes should be comparatively narrower than those for the mathematical modes. Fig. 1 shows the stabilization diagram [1] obtained by running PTD algorithm. The stabilization diagram also shows the confidence intervals, albeit only for the stabilized modes (indicated as blue diamonds ◊ in the stabilization diagram).

It can be observed from the stabilization diagram in Fig. 1 that confidence intervals for the actual system poles (for e.g. modes at 34.86 Hz and 104.77 Hz) are expectedly much narrower in comparison to the confidence intervals of the mathematical poles (mode 120 Hz in iteration 21). This behavior can be observed more clearly by zooming around 12.53 Hz mode (See Fig. 2) where there are few mathematical modes (as indicated in Fig. 2) that stabilize in close vicinity of the actual system pole. It is easily observable that these mathematical poles have much wider confidence interval in comparison to the actual system mode. This provides an extra means for distinguishing mathematical poles from the actual structural modes. The confidence interval related information can also be used to clean the stabilization diagram as that can ease the process of mode selection (from the stabilization diagram) for the user.

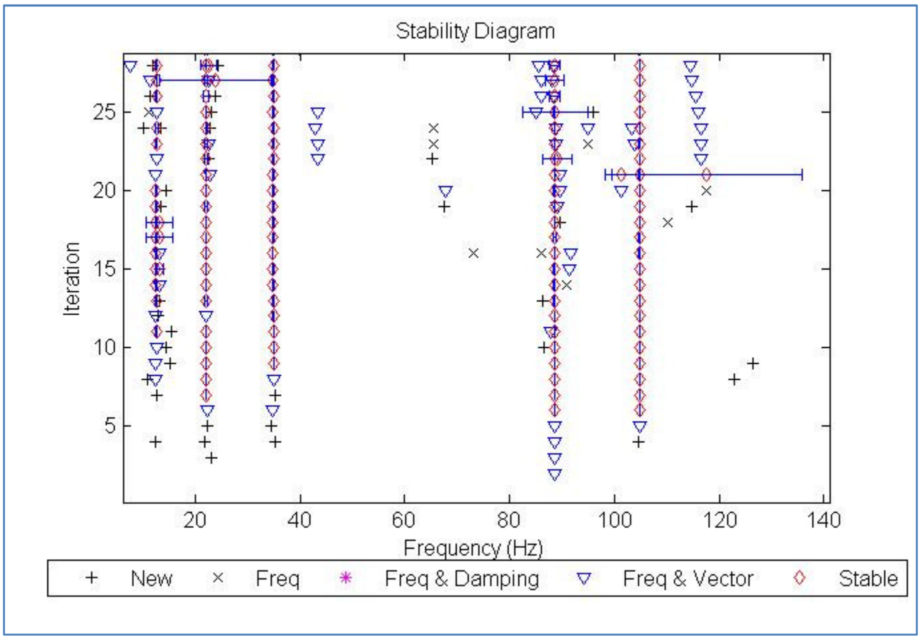

Figure 1: Stabilization Diagram with Confidence Intervals

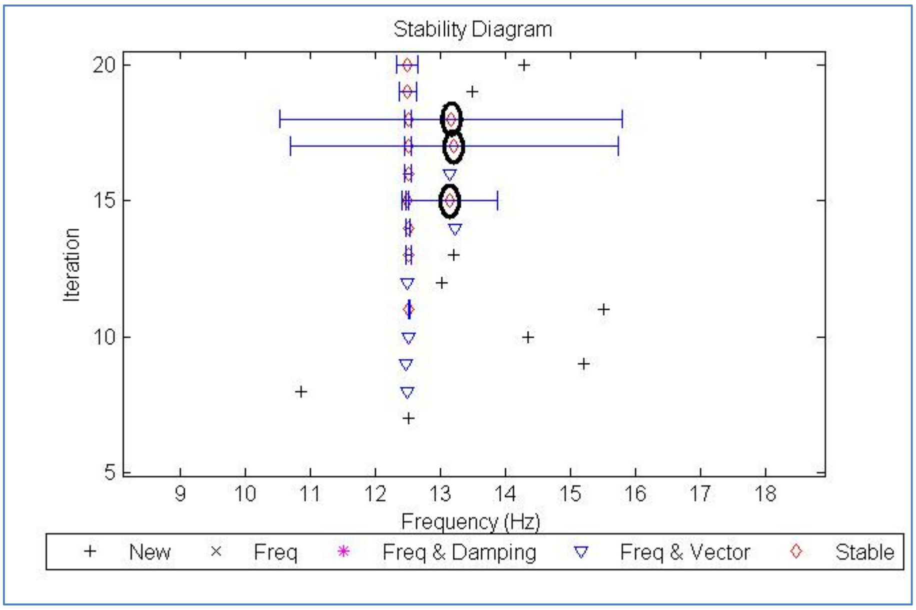

Figure 2: Section of Stabilization Diagram zoomed around 12.53 Hz mode

4. Conclusions

This paper presents a theoretical framework to quantify uncertainty associated with high order based modal parameter estimation procedure by calculating confidence intervals of the estimated modal parameters. The approach is validated by means of Monte Carlo simulations conducted on a 5 degrees-of-freedom analytical system. It is shown that confidence intervals obtained using the proposed theory matches well with those from the sample statistics based on the Monte Carlo simulations.

It is also illustrated in the paper how the knowledge of confidence intervals of the modal parameters (in other words quantified uncertainty of parameter estimation procedure) can provide more insights regarding the quality of the estimated parameters and can act as a valuable tool that can help user to make a better choice of modal parameters representing the dynamics of the structure.

Acknowledgements

The author would like to thank Prof. Jerome Antoni, University of Lyon, for his valuable inputs while undertaking the research summarized in this paper.

References

1. Allemang, R.J., “Vibrations: Experimental Modal Analysis”, Structural Dynamics Research Laboratory, Department of Mechanical, Industrial and Nuclear Engineering, University of Cincinnati, 1999. http://www.sdrl.uc.edu/academic-course-info/vibrations-iii-20-263-663/
2. Pintelon, R., Guillaume, P., Schoukens, J., “Uncertainty calculation in (operational) modal analysis”, Mechanical Systems and Signal Processing, Volume 21, pp. 2359-2373, 2007.
3. Reynders, E., Pintelon, R., De Roeck, G., “Uncertainty bounds on modal parameters obtained from stochastic subspace identification”, Mechanical Systems and Signal Processing, Volume 22, pp. 948-969, 2008.
4. Brincker, R., Andersen, P., “Understanding stochastic subspace identification”, Proceedings of the 24th International Modal Analysis Conference, St. Louis (MO), USA, 2006.
5. Dohler, M., Lam, X.B., Mevel, L., “Uncertainty quantification for modal parameters from stochastic subspace identification on multi-setup measurements”, Mechanical Systems and Signal Processing, Volume 36, pp. 562-581, 2013.
6. Rees, D.G., Essential Statistics, Fourth Edition, Chapman and Hall/CRC Press, Florida, 2000.
7. Vold, H., Rocklin, T.; “The numerical implementation of a multi-input modal estimation algorithm for mini-computers”; Proceedings of the 1st IMAC, Orlando, FL, November 1982.
8. Vold, H., Kundrat, J., Rocklin, T., Russell, R.; “A multi-input modal estimation algorithm for mini-computers”, SAE Transactions, Volume 91, Number 1, pp. 815-821, January, 1982.
9. Greene, W., Econometric Analysis, Seventh edition, Prentice Hall, Upper Saddle River, NJ, 2011.
10. Van De Geer, S.A., “Least Squares Estimation”, Encyclopedia of Statistics in Behavioral Science, Volume 2, pp. 1041-1045, John Wiley and Sons, Chichester, 2005.
11. Lutkepohl, H.; New Introduction to Multiple Time Series Analysis, Springer, Germany, 2005.
12. Agnew, D.C., Constable, C., “Least Squares Estimation”, Geophysical Data Analysis Course Notes, Institute of Geophysics and Planetary Physics, University of California San Diego, 2008. http://igpphome.ucsd.edu/~cathy/Classes/SIO223A/sio223a.chap7.pdf
13. Navidi, W., Statistics for Engineers and Scientists, 3rd Edition, McGraw Hill, NY, 2010.
14. Taylor, J.R., An Introduction to Error Analysis, 2nd Edition, University Science Books, 1997.