A Non-Hermitian Joint Diagonalization based Blind Source Separation algorithm for Operational Modal Analysis

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Nomenclature

$\mathbf{x}$	State vector
$\mathbf{A_d}$	State matrix
$\mathbf{y}$	Measurement vector
$\mathbf{f}$	Unknown force vector
$\boldsymbol{\upsilon}$	Measurement noise vector
λ_i	i^{th} pole of the system
T_s	Sampling Period
$\mathbf{\Phi}$	Mode shapes of the system
$\mathbf{R_{yy}}$	Correlation matrix of the measurements
τ	Time lag
$\mathbf{H}_{yy}$	Hankel matrix of corrleations
$\hat{\mathbf{L}}$, $\hat{\mathbf{R}}$	Left and Right factors of Hankel matrix, containing information about mode shape.
$\mathbf{\Sigma}^{\tau}$	Diagonal matrix such that $\mathbf{H}_{yy}[\tau] = \hat{\mathbf{L}}\mathbf{\Sigma}^{\tau}\hat{\mathbf{R}}$, contains information about system poles.

ABSTRACT

Second Order Blind Source Separation (SO-BSS) techniques possess several mathematical characteristics making them a viable option for Operational Modal Analysis (OMA). However, on closer scrutiny it is revealed that there are certain subtleties that limit their direct application to OMA applications. This paper continues from past work of the authors, which focussed on understanding SO-BSS techniques from a perspective of OMA applicability and developing SO-BSS based algorithm for OMA. In this paper, a new algorithm is proposed that overcomes the inherent limitations of SO-BSS algorithms with regards to their applicability to OMA. These limitations include applicability to heavily damped systems, identification of complex modes, and applicability to scenarios where number of available sensors is lesser than the number of modes to be estimated, etc. The algorithm's advantage over original form of SO-BSS is demonstrated by means of an analytical example.

1. Introduction

This paper is continuation of authors' effort towards understanding and utilizing second order blind source separation (SOBSS) techniques for operational modal analysis (OMA) [1, 2]. SOBSS algorithms have obvious appeal in OMA domain since both of them share fundamental mathematical similarities. Several recent works [3-9] have explored the possibility of utilizing SOBSS algorithms for OMA to varying degrees of success. The success of SOBSS algorithms in context of OMA has been limited by several factors including the inability of SOBSS algorithms to estimate complex modes, heavily damped systems, estimate more modes than number of sensors etc., in their original form. In [1] authors have explained mathematical theory behind SOBSS algorithms in terms of OMA framework and showed how they are related to Stochastic Subspace Identification (SSI) algorithm [10, 11]. Based on this knowledge, authors proposed an Alternating Least Squares (ALS) based Parallel Factorization (PFA) algorithm in [2]. It was shown by means of a simulated system that this algorithm is capable of estimating heavily damped modes and complex mode shapes. However, one of the concerns regarding this proposed algorithm was its convergence and developing a more robust algorithm with better convergence properties was suggested as a future step in this research.

This paper proposes a new SOBSS based OMA algorithm that overcomes the aforementioned inherent limitations of SOBSS techniques. Proposed algorithm generalizes SOBSS to estimate complex mode shapes, and to handle more active modes than the number of available sensors, as is often encountered in practice. This algorithm is based on Non-Hermitian factorization of the covariance matrices and is termed Non-Hermitian Joint Approximate Diagonalization (NoHeJAD).

The paper is organized in following manner: Section 2 briefly recalls the previous work carried out in [1]. Firstly, mathematical fundamentals of SOBSS techniques are understood within OMA framework and then, in the light of this knowledge, various limitations regarding their application to OMA are discussed. NoHeJAD algorithm is proposed in Section 3. This section also explains how NoHeJAD algorithm overcomes the limitations discussed in section 2. Application of NoHeJAD algorithm is illustrated by means of a simple analytical system in Section 4, followed by discussions and conclusions.

2. Understanding SOBSS within OMA Framework

The matrix differential equation of motion of an n-degree-of-freedom system is given by

$$\mathbf{M}\,\ddot{\mathbf{x}}(t)+\mathbf{C}\,\dot{\mathbf{x}}(t)+\mathbf{K}\,\mathbf{x}(t)=\mathbf{f}(t) \qquad (01)$$

with mass matrix $\mathbf{M}\in [\ ]^{n\times n}$, stiffness matrix $\mathbf{C}\in [\ ]^{n\times n}$, damping matrix $\mathbf{K}\in [\ ]^{n\times n}$, and forcing vector $\mathbf{f}(t)\in \{\ \}^{n}$ which entails the displacement vector $\mathbf{x}(t)\in \{\ \}^{n}$. In practice, the dynamics of the system is observed through measurements,

$$\mathbf{y}(t)=\mathbf{C}^{i}\,\mathbf{x}^{(i)}(t)+\mathbf{v}(t),\in \{\ \}^{m}, \qquad (02)$$

where i = 0,1,2 depends on whether response is represented in terms of displacement, velocity or acceleration, $\mathbf{C}^{i}\in [\ ]^{m\times n}$ is the measurement matrix, and $\mathbf{v}(t)$ is a vector of measurement noise.

The objective of OMA is to recover the modal parameters of the system, i.e. the poles λ_i and the eigenvectors ϕ_i, from the observations $\mathbf{y}(t)$ only. Specifically, for an n-degree-of-freedom system, the unknowns of the problem are the diagonal matrix $\mathbf{\Lambda}=diag(\lambda_1,\cdots\cdots,\lambda_n)\in \mathrm{C}^{m\times n}$ of poles (for simplicity it will be assumed that all poles are distinct, i.e. $\lambda_i \neq \lambda_j, i\neq j$) and the modal matrix $\mathbf{\Phi}=[\phi_1,\cdots\cdots,\phi_n]\in \mathrm{C}^{n\times n}$ of eigenvectors such that $(\lambda_i^2\mathbf{M}+\lambda_i\mathbf{C}+\mathbf{K})\,\phi_i=0$, for any *i*.

There are certain key assumptions which are made while applying OMA techniques. These assumptions are listed below.

- H1: $\mathbf{f}(t)$ and $\mathbf{v}(t)$ are two zero-mean random stationary processes,
- H2: $\mathbf{f}(t)$ and $\mathbf{v}(t)$ are mutually uncorrelated,
- H3: $\mathbf{f}(t)$ and $\mathbf{v}(t)$ have flat spectra in comparison to the resonances of the system in the frequency range of interest.
- H4: Forces $\mathbf{f}(t)$ are randomly distributed at all n degrees-of-freedom.

The last assumption does not have any direct implication with respect to the work presented, but is significant in context of OMA as it ensures that all the modes in the frequency range of interest are sufficiently excited.

It is now well established that the key to applying SOBSS techniques for OMA lies in the ability of these algorithms to decompose the measured output responses to corresponding modal coordinates and modal filters according to the expansion theorem [12] in the following manner.

$$\mathbf{x}(t) = \mathbf{\Phi}\boldsymbol{\eta}(t), \qquad (03)$$

where $\boldsymbol{\eta}(t) \in \{\ \}^n$ denote the vector of modal coordinates related to the set of active mode filters $\{\phi_i\}$, i =1,...,n. Once this decomposition is achieved, modal frequencies and damping information is obtained from modal coordinates and modal filters are nothing but mode shapes. Similarity of Eqn. (03) to mathematical form of a typical BSS model makes it easier to understand the obvious application of SOBSS algorithms for OMA purposes. In practice, vector x(t) must also be replaced by measurement vector y(t) of Eqn. (02); the resemblance with the BSS mixture model then still holds, provided that

- H5: there are as many measurements as sources, $n \le m$,
- H6: the additive noise $\mathbf{v}(t)$ is negligible or can be filtered out, for instance by subspace techniques [13].

Under these assumptions, the correlation matrix of the (discrete-time) measurements is given by

$$\mathbf{R}_{yy}[\tau] = \mathbf{\Phi}\,\mathbf{R}_{yy}[\tau]\,\mathbf{\Phi}^H, \text{ at any } \tau, \qquad (04)$$

The identification of the modal matrix $\mathbf{\Phi}$ and then separate modal coordinates $\eta_i(t)$ using SOBSS algorithms, such as AMUSE [14] and SOBI [15], typically involves joint diagonalization of the response correlation matrices. As mentioned earlier, feasibility of SOBSS for OMA has been reported and illustrated in several recent publications. In [1] it was shown that the correlation matrix of the measurements has expression

$$\mathbf{R}_{yy}[\tau] = \mathbf{L}\mathbf{\Sigma}_d^{\tau}\mathbf{R} + \mathbf{L}\mathbf{\Sigma}_d^{\tau-1}\mathbf{B}_d\mathbf{R}_{ff}[\tau]\mathbf{D}_d^{iH} + \mathbf{R}_{vv}[\tau], \tau > 0 \qquad (05)$$

with

$$\mathbf{L} = \mathbf{C}_d^i\mathbf{\Psi} \text{ and } \mathbf{R} = \mathbf{R}_{qq}[0]\mathbf{L}^H, \qquad (06)$$

and with $\mathbf{R}_{qq}[0]$, $\mathbf{R}_{ff}[\tau]$ and $\mathbf{R}_{vv}[\tau]$ the correlation matrices of the state-space modal coordinates, the force and the noise, respectively, $\mathbf{\Sigma}_d$ is diagonal matrix of eigenvalues having modal frequency and damping information and $\mathbf{\Psi}$ is state-space modal matrix. This expression, under assumptions H1, H2 and H3 reduces to

$$\mathbf{R}_{yy}[\tau] = \mathbf{L}\mathbf{\Sigma}_d^{\tau}\mathbf{R}, \tau \ge \tau_0, \qquad (07)$$

for some $\tau_0 > 0$. Using the fact that $\mathbf{R}_{qq}[\tau] = \mathbf{\Sigma}^{\tau}\mathbf{R}_{qq}[0]$, where $\mathbf{R}_{qq}[0]$ is the (unknown) correlation matrix of the modal coordinate $\mathbf{q}[k] = \mathbf{\Psi}^{-1}\mathbf{x}[k]$, Eq. (07) can finally be rearranged as

$$\mathbf{R}_{yy}[\tau] = \mathbf{L}\mathbf{R}_{qq}[\tau]\mathbf{L}^H \,, \tau \geq \tau_0 \,, \tag{08}$$

which is the usual form, symmetrical in the left and right factors, that enters SOBSS algorithms. Indeed, this is similar to Eqn. (04). Therefore, for SOBSS to correctly estimate factor $\mathbf{L}$ as the joint diagonaliser of the correlation matrix $\mathbf{R}_{yy}[\tau]$ at time lags $\tau \geq \tau_0$, $\mathbf{R}_{qq}[\tau]$ must be a diagonal matrix. For this to happen, the modal coordinates should have nearly disjoint spectra in the frequency domain (uncoupled resonances) or, in other words, are approximately uncorrelated in the time domain [4, 5]. Physically for an arbitrary loading, this is the case when, and only when, the damping of the system is light and the system does not have closely spaced modes. Thus, due to the mathematical formulation expressed by Eqn. (08), SOBI and AMUSE typically estimate *real* mode shapes, approximating those of the underlying undamped system. This mathematical formulation also puts a constraint that there should be at least as many sensors as the number of modes to be identified.

3. Non-Hermitian Joint Approximate Diagonalization (NoHeJAD) Algorithm

As described in previous section, SOBSS algorithms suffer from various shortcomings when being utilized in their inherent form for OMA. Thus they need to be modified in order to make them compliable with OMA requirements. This section first lays down mathematical foundations of NoHeJAD algorithm; explains how it mathematically overcomes the shortcomings of SOBSS algorithms; and then describes the implementation of NoHeJAD.

The basis of NoHeJAD algorithm is that it searches for the left and right factors of the correlation matrix of measurements, $\mathbf{R}_{yy}[\tau]$, in the non-Hermitian form as given by Eqn. (07) instead of the usual form in Eqn. (08), which is typical to most SOBSS algorithms. For the sake of simplicity, it is assumed that m = 2n, i.e. number of sensors are equal to twice the number of degrees of freedom or in other words the number of modes of the system. In such a case, there exist unique inverses, $\mathbf{L}^+ \in [\ \ ]^{2n\times 2n}$ and $\mathbf{R}^+ \in [\ \ ]^{2n\times 2n}$, to matrices $\mathbf{L}$ and $\mathbf{R}$ in Eqn. (07), such that after pre and post-multiplication of Eqn. (07) with $\mathbf{L}^+$ and $\mathbf{R}^+$, one obtains

$$\mathbf{L}^+\mathbf{R}_{yy}[\tau]\mathbf{R}^+ = \underbrace{\mathbf{L}^+\mathbf{L}}_{\mathbf{I}_{2n}}\boldsymbol{\Sigma}_d^\tau \underbrace{\mathbf{R}\mathbf{R}^+}_{\mathbf{I}_{2n}} = \boldsymbol{\Sigma}_d^\tau \,, \tau \geq \tau_0 \tag{09}$$

Keeping in mind that $\boldsymbol{\Sigma}$ is in general a diagonal matrix this means that $\mathbf{L}^+$ and $\mathbf{R}^+$ are joint diagonalisers of the set of matrices $\{\mathbf{R}_{yy}[\tau]\}_{\tau\geq\tau_0}$. Therefore, an immediate generalisation of AMUSE/SOBI for OMA is to seek for a couple of factors, $\mathbf{L}^+$ and $\mathbf{R}^+$, which minimise the cost function

$$J_{\mathrm{Off}}(\mathbf{L}^+, \mathbf{R}^+) = \sum_{\tau\in\mathrm{T}} w_\tau . \left\| \mathrm{Off}(\mathbf{L}^+\mathbf{R}_{yy}[\tau]\mathbf{R}^+) \right\|^2 \tag{10}$$

for a given set of weights $\{w_\tau\}_{\tau\in\mathrm{T}}$ and subject to some constraint that prevents the trivial solutions $\mathbf{L}^+ = 0$ or $\mathbf{R}^+ = 0$. Specifically, it is proposed herein to minimise the magnitude of off-diagonal elements whilst at the same time forcing non-zero diagonal elements, that is minimise Eqn. (10) subject to constraint

$$J_{\mathrm{Diag}}(\mathbf{L}^+, \mathbf{R}^+) = \sum_{\tau\in\mathrm{T}} w_\tau . \left\| \mathrm{Diag}(\mathbf{L}^+\mathbf{R}_{yy}[\tau]\mathbf{R}^+) \right\|^2 = C > 0 \,, \tag{11}$$

for some constant C, where $\mathrm{Diag}(\bullet)$ zeroes the off-diagonal elements of a matrix. Once the joint diagonalisers are estimated, the modal matrix $\mathbf{L}$ is simply returned by the inverse of $\mathbf{L}^+$. Additionally, $\mathrm{Diag}(\mathbf{L}^+\mathbf{R}_{yy}[\tau]\mathbf{R}^+)$ returns an estimate of $\boldsymbol{\Sigma}^\tau$ which contains the poles of the system that can be estimated using any standard SDOF method.

It is important to notice that in this formulation left and right diagonalisers of the correlation matrix $\mathbf{R}_{yy}[\tau]$ are not constrained to be Hermitian transforms of each other, i.e. $\mathbf{L}^{+} \neq \mathbf{R}^{+H}$. This allows more flexibility in trying to jointly diagonalise $\mathbf{R}_{yy}[\tau]$, which is the key property that makes possible the recovery of complex mode shapes as this approach guarantees that a solution exists in the most general sense, independently of the degree of damping. Further since the whitening step (computation of and division by the square-root of $\mathbf{R}_{yy}[0]$), is avoided, the suggested approach is more robust against additive noise (see [1] for more discussion on the influence of additive noise on $\mathbf{R}_{yy}[0]$).

The Non-Hermitian Joint Approximate Diagonalization (NoHeJAD) algorithm proposed in this paper solves the minimization problem in Eqn. (11). However, before describing the algorithm, the interesting case of having less sensors than the number of modes of interest is discussed. In these regard, a simple approach can be the one suggested in [9].

As explained earlier, the suggested approach of estimating $\mathbf{L}^{+}$ and $\mathbf{R}^{+}$ eliminates the pre-whitening step thus making it possible for the algorithm to operate directly on correlation functions estimated after processing the acquired output time histories. This serves two purposes: firstly, signal processing techniques such as windowing and averaging can be used to calculate the correlation functions through power spectra and improve the performance of the algorithm in situations when output time histories are contaminated with noise and don't have a high SNR. Additionally, since the algorithm operates on correlation functions, one can now select a specific frequency range to apply this algorithm. As a consequence the problem of having more modes than number of sensors available can be tackled effectively by dividing the entire frequency range of interest into smaller ranges (such that number of modes in the selected range are less than the number of sensors) and then apply NoHeJAD one by one to them. This is different in comparison to typical SOBSS algorithms, like SOBI, that are applied in the entire frequency range. These aspects are covered in more details in [9], where authors modified original SOBI formulation so that it can operate on correlation functions, making it possible to incorporate signal processing techniques and extracting modal parameters within specified frequency ranges.

Although one can take the above mentioned approach to deal with the case when one has less sensors than the number of modes and utilize the NoHeJAD in its basic form, it is also possible to generalize NoHeJAD such that it can take care of this scenario by itself. The key lies in utilizing the Hankel matrix of correlations.

$$\mathbf{H}_{yy}[\tau] = \begin{bmatrix} \mathbf{R}_{yy}[\tau] & \beta_1 \mathbf{R}_{yy}[\tau+1] & \cdots & \beta_{K-1}\mathbf{R}_{yy}[\tau+K-1] \\ \beta_1 \mathbf{R}_{yy}[\tau+1] & \beta_1^2 \mathbf{R}_{yy}[\tau+2] & \cdots & \beta_1 \beta_{K-1}\mathbf{R}_{yy}[\tau+K] \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{K-1}\mathbf{R}_{yy}[\tau+K-1] & \beta_{K-1}\beta_1 \mathbf{R}_{yy}[\tau+K] & \cdots & \beta_{K-1}^2 \mathbf{R}_{yy}[\tau+2K-2] \end{bmatrix}, \tag{12}$$

with dimension $mK \times mK$, where $\{\beta_i > 0;\ i = 1,2,\ldots.. K - 1\}$ is a sequence of user-defined weights. From Eqn. (07), the Hankel correlation matrix factorises as

$$\mathbf{H}_{yy}[\tau] = \begin{bmatrix} \mathbf{L} \\ \beta_1 \mathbf{L\Sigma} \\ \vdots \\ \beta_{K-1}\mathbf{L\Sigma}^{K-1} \end{bmatrix} \mathbf{\Sigma}^{\tau} \left[\mathbf{R} \quad \beta_1 \mathbf{R\Sigma} \quad \cdots \quad \beta_{K-1}\mathbf{R\Sigma}^{K-1} \right] = \hat{\mathbf{L}}\mathbf{\Sigma}^{\tau}\hat{\mathbf{R}}, \quad \tau \geq \tau_0 \tag{13}$$

with $\hat{\mathbf{L}} \in [\ ]^{mK\times 2n}$ and $\hat{\mathbf{R}} \in [\ ]^{2n\times mK}$. This is exactly the same form as in Eqn. (07), with the so-called observability matrix $\hat{\mathbf{L}}$ substituted for $\mathbf{L}$, and controllability matrix $\hat{\mathbf{R}}$ for $\mathbf{R}$. Therefore, provided that K is set large enough so that $mK > 2n$, the same algorithm can be used to find the couple of left and right pseudo inverses, $\hat{\mathbf{L}}^{+}$ and $\hat{\mathbf{R}}^{+}$, which jointly diagonalises the Hankel correlation matrices $\mathbf{H}_{yy}[\tau]$ for a set of time-lags. The mode shape information is contained within first ($m \times 2n$) block of $\hat{\mathbf{L}}$ and modal frequencies and damping can be obtained in the similar manner as described for ($m = 2n$) case.

Based on this understanding, a simple algorithm, with good convergence properties, is presented here to solve the minimization problem in Eqn. (11). This algorithm is coined Non-Hermitian Joint Approximate Diagonalization (NoHeJAD) algorithm and is partly inspired by the recent work reported in [16]. The goal of the algorithm is to minimise Eqns (10) and (11). One Key aspect with regards to implementing this algorithm is that matrices $\mathbf{L}^{+}$ and $\mathbf{R}^{+}$ have specific polarities that are to be preserved right from the onset. Indeed, it is seen that i) $\mathbf{L}_{:,1:n} = \mathbf{L}^{*}_{:,n+1:2n}$, where $\mathbf{L}_{:,1:n}$ and $\mathbf{L}_{:,n+1:2n}$ are the left and right blocks in matrix $\mathbf{L}$, and ii) $\mathbf{R}_{1:n,:} = \mathbf{R}^{*}_{n+1:2n,:}$, where $\mathbf{R}_{1:n,:}$ and $\mathbf{R}_{n+1:2n,:}$ are the top and bottom blocks in the Hankel matrix formed using output covariances $\mathbf{R}_{yy}[\tau]$. Therefore, one must similarly have iii) $\mathbf{L}^{+}_{1:n,:} = \mathbf{L}^{+*}_{n+1:2n,:}$ and iv) $\mathbf{R}^{+}_{:,1:n} = \mathbf{R}^{+*}_{:n+1:2n}$.

If $\mathbf{l}^{i}$ and $\mathbf{r}^{i}$ are i[th] row and column of matrices $\mathbf{L}^{+}$ and $\mathbf{R}^{+}$ respectively, then Eqn. (10) can be expanded either as

$$J_{\text{Off}}(\mathbf{L}^{+},\mathbf{R}^{+}) = \sum_{i=1}^{2n} \mathbf{l}^{i}\left(\sum_{\tau=\mathrm{T}} \mathbf{R}_{yy}[\tau](\mathbf{R}^{+}\mathbf{R}^{+H} - \mathbf{r}^{i}\mathbf{r}^{iH})\right).\mathbf{l}^{iH} = \sum_{i=1}^{2n} \mathbf{l}^{i}\mathrm{M}^{i}_{\text{Off}}(\mathbf{R}^{+}).\mathbf{l}^{iH} \tag{12}$$

or equivalently as

$$J_{\text{Off}}(\mathbf{L}^{+},\mathbf{R}^{+}) = \sum_{j=1}^{2n} \mathbf{r}^{jH}\left(\sum_{\tau=\mathrm{T}} \mathbf{R}_{yy}[\tau](\mathbf{L}^{+H}\mathbf{L}^{+} - \mathbf{l}^{jH}\mathbf{l}^{j})\right).\mathbf{r}^{j} = \sum_{j=1}^{2n} \mathbf{r}^{jH}\mathrm{M}^{j}_{\text{Off}}(\mathbf{L}^{+}).\mathbf{r}^{j}\,. \tag{13}$$

where $\mathrm{M}^{i}_{\text{Off}}(\bullet) = \mathrm{M}^{i}_{\text{Off}}(\bullet)^{H}$ is a Hermitian matrix.

Eqn. (11) can also be expanded on similar lines as following,

$$J_{\text{Diag}}(\mathbf{L}^{+},\mathbf{R}^{+}) = \sum_{i=1}^{2n} \mathbf{l}^{i}\left(\sum_{\tau=\mathrm{T}} \mathbf{R}_{yy}[\tau](\mathbf{R}^{+}\mathbf{R}^{+H} - \mathbf{r}^{i}\mathbf{r}^{iH})\right).\mathbf{l}^{iH} = \sum_{i=1}^{2n} \mathbf{l}^{i}\mathrm{M}^{i}_{\text{Diag}}(\mathbf{R}^{+}).\mathbf{l}^{iH} \tag{14}$$

or

$$J_{\text{Diag}}(\mathbf{L}^{+},\mathbf{R}^{+}) = \sum_{j=1}^{2n} \mathbf{r}^{jH}\left(\sum_{\tau=\mathrm{T}} \mathbf{R}_{yy}[\tau](\mathbf{L}^{+H}\mathbf{L}^{+} - \mathbf{l}^{jH}\mathbf{l}^{j})\right).\mathbf{r}^{j} = \sum_{j=1}^{2n} \mathbf{r}^{jH}\mathrm{M}^{j}_{\text{Diag}}(\mathbf{L}^{+}).\mathbf{r}^{j}\,. \tag{15}$$

where $\mathrm{M}^{i}_{\text{Diag}}(\bullet) = \mathrm{M}^{i}_{\text{Diag}}(\bullet)^{H}$ is another Hermitian matrix. These equations suggest the alternate minimisation of Eqns. (10) and (11) with respect to $\mathbf{l}^{i}$ with $\mathbf{R}^{+}$ fixed and with respect to $\mathbf{r}^{i}$ with $\mathbf{L}^{+}$ fixed. Eqns. (12-15) can be reformulated as local optimization problem as

$$\begin{aligned}
\mathbf{l}^{i}_{(n)} &= \arg\min_{\mathbf{l}^{i}}\left\{\sum_{j=1}^{2n} \mathbf{l}^{j}\mathrm{M}^{j}_{\text{Off}}(\mathbf{R}^{+}_{(n-1)})\mathbf{l}^{jH}\right\} + \mu_{i,1}\left(C - \sum_{j=1}^{2n} \mathbf{l}^{j}\mathrm{M}^{j}_{\text{Diag}}(\mathbf{R}^{+}_{(n-1)})\mathbf{l}^{jH}\right) \\
\mathbf{r}^{i}_{(n)} &= \arg\min_{\mathbf{r}^{i}}\left\{\sum_{j=1}^{2n} \mathbf{r}^{jH}\mathrm{M}^{j}_{\text{Off}}(\mathbf{L}^{+}_{(n)})\mathbf{r}^{j}\right\} + \mu_{i,2}\left(C - \sum_{j=1}^{2n} \mathbf{r}^{jH}\mathrm{M}^{j}_{\text{Diag}}(\mathbf{L}^{+}_{(n)})\mathbf{r}^{j}\right)
\end{aligned} \tag{16}$$

where $\mu_{i,1}$ and $\mu_{i,2}$ are Lagrange multipliers.

It is important while initializing this optimization procedure that $\mathbf{R}^{+}_{(0)}$ is initialized with complex values fulfilling the polarity structure (iv). From standard optimization theory, the solutions to Eqn. (16) are eigenvectors associated with the least eigenvalues of the generalized eigenvalue problems

$$\begin{cases} \mathrm{M}^{j}_{\mathrm{Off}}(\mathbf{R}^{+}_{(n-1)})\mathbf{l}^{jH}_{(n)} = \mu_{i,1}\,\mathrm{M}^{j}_{\mathrm{Diag}}(\mathbf{R}^{+}_{(n-1)})\mathbf{l}^{jH}_{(n)} \\ \mathrm{M}^{j}_{\mathrm{Off}}(\mathbf{L}^{+}_{(n)})\mathbf{r}^{j}_{(n)} = \mu_{i,2}\,\mathrm{M}^{j}_{\mathrm{Diag}}(\mathbf{L}^{+}_{(n)})\mathbf{r}^{j}_{(n)} \end{cases} \tag{17}$$

Note that because $\mathrm{M}^{i}_{\mathrm{Off}}(\bullet)$, $\mathrm{M}^{i}_{\mathrm{Diag}}(\bullet)$ are Hemitian matrices, the associated eigenvalues are necessarily non-negative. Moreover, a necessary condition for the eigenvalue problems in Eqn. (17) to have a unique solution is to enforce $\mathrm{M}^{i}_{\mathrm{Off}}(\bullet)$ and $\mathrm{M}^{i}_{\mathrm{Diag}}(\bullet)$ with full rank. This can only be achieved if the number of summed time-lags in Eqs. (12)-(15) is not less than the dimension of the correlation matrix, i.e. if $|\mathrm{T}| \geq m$. Finally, it should be recognised that the polarity properties (iii) and (iv) imply that $\mathbf{l}^{i} = (\mathbf{l}^{n+i})^{*}$ and $\mathbf{r}^{i} = (\mathbf{r}^{n+i})^{*}$ for $i = 1,\ldots,n$, so that Eqn. (17) has to be solved for $\mathrm{i} \leq n$ only.

4. Example: 15 DOF Analytical System

An analytical 15 degrees of freedom system is considered in this section to demonstrate the applicability of NoHeJAD. This system simulates complex, heavily damped, and strongly coupled modes and is typical of system possessing characteristics that have been found to push SOBSS algorithms to their limits. The system is excited by white Gaussian forces having unity magnitude and random phase and its acceleration responses are available at all degrees of freedom (m = 15). All signals are sampled at 1024 Hz and the correlation matrix is computed on 163840-sample-long records.

Fig. 1 shows the power spectra of the output response at all 15 DOFs along with modal coordinates separated by NoHeJAD, which was run for 100 iterations.

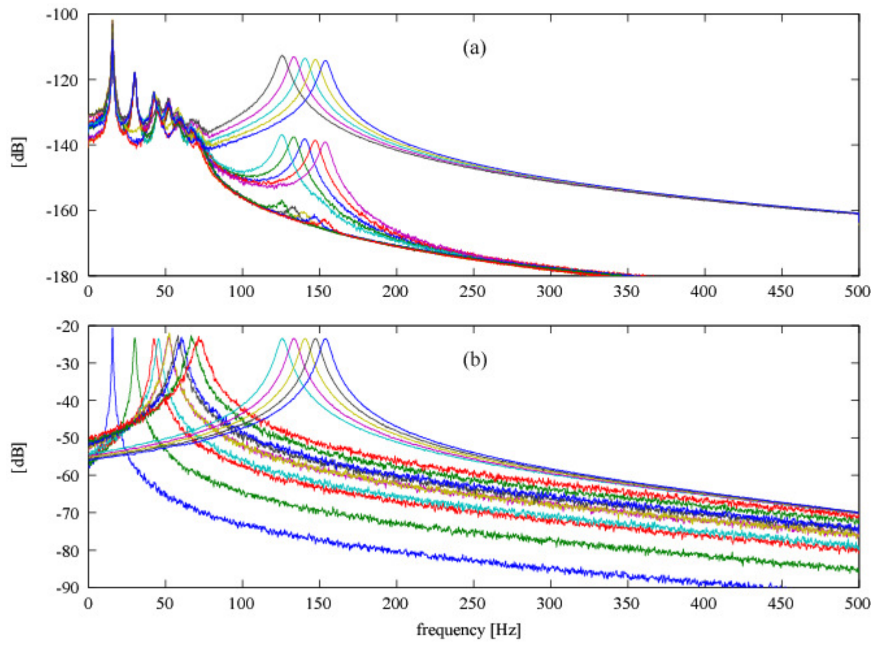

Fig. 1: Power spectra of **a)** the system responses and **b)** the separated modal coordinates by NoHeJAD (frequency resolution Δf = 0.4Hz)

Fig. 2 displays the evolution of $J_{\mathrm{Off}}(\mathbf{L}^{+},\mathbf{R}^{+})\big/J_{\mathrm{Diag}}(\mathbf{L}^{+},\mathbf{R}^{+})$ as a function of iterations, which shows quite a fast convergence (a tenth of iterations) of the algorithm. The resulting joint approximation diagonalisation is appraised by means of the ratio

$$\frac{\left\|\mathrm{Off}(\mathbf{L}^{+}\mathbf{R}_{yy}[\tau]\mathbf{R}^{+})\right\|^{2}}{\left\|\mathrm{Diag}(\mathbf{L}^{+}\mathbf{R}_{yy}[\tau]\mathbf{R}^{+})\right\|^{2}} \tag{12}$$

as a function of τ in Fig. 3. This function also reflects the signal-to-noise ratio in the correlation matrix which slightly increases with time-lags as a collateral effect of the exponential decrease of the system impulse response.

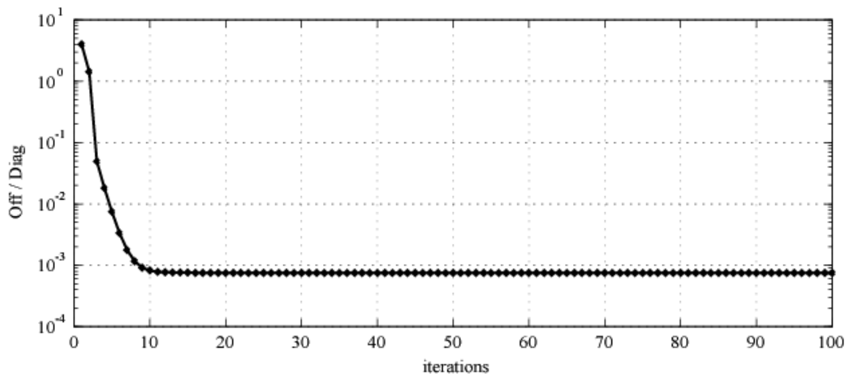

Fig. 2: Convergence of the NoHeJAD algorithm

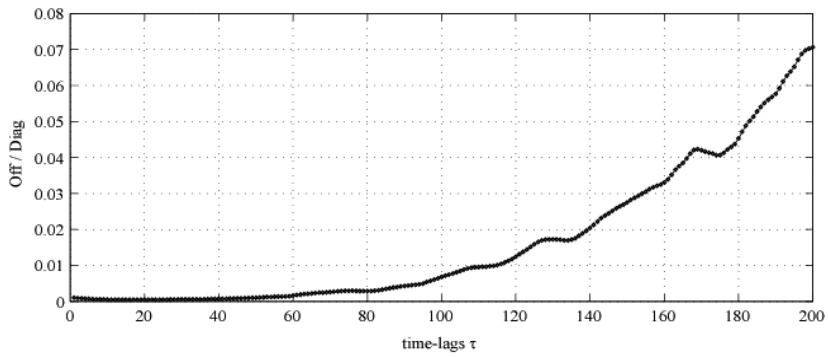

Fig. 3: Joint diagonalisation assessment as a function of time-lags

The estimated modal parameters and related estimation errors are reported in Tables 1 and 2, together with a comparison of the estimates returned by SOBI. It is seen that SOBI and NoHeJAD perform similarly well to estimate the natural frequencies and damping of the system in this case; this reflects the robustness of SOBI to operate even when the assumption of a conservative system is not perfectly fulfilled. Table 3 displays the modal assurance criteria (MAC) on the real and imaginary parts of estimated mode shapes from. It is observed here that SOBI is unable to estimate the imaginary parts of the mode shapes, as explained in section 2. It is worth pointing here that the smaller the imaginary parts of the mode shapes relatively of their real parts, the more difficult their estimation; this explains some of the low MAC values in Table 3. Finally, Fig. 1(b) displays the power spectra of separated modal coordinates computed by taking the Fourier transforms of the elements in $\mathrm{Diag}(\mathbf{L}^{+}\mathbf{R}_{yy}[\tau]\mathbf{R}^{+})$. This visually proves the excellent separation capability of the proposed algorithm. The power spectrum of the i-th separated modal coordinate may also be obtained from the cross-spectrum between signals $\mathrm{L}_i^{+};\mathbf{y}_a(t)$ and

$R_i^{+H};\mathbf{y}_a(t)$, where $\mathbf{y}_a(t)=[y_1(t),\cdots\cdots,y_m(t),\cdots\cdots,y_1(t+K),\cdots\cdots,y_m(t+K)]^T$, that is after application of two different modal filters on the measured responses.

Table 1: Estimated natural frequencies

Natural frequencies f_n (Hz)		
True	**SOBI**	**NoHeJAD**
15.99	15.99	15.99
30.86	30.82	30.82
43.60	43.62	43.62
46.44	46.49	46.49
53.32	53.32	53.32
53.39	53.50	53.51
59.41	59.46	59.44
61.62	61.73	61.73
68.81	68.86	68.88
73.63	73.61	73.64
128.84	128.87	128.87
136.55	136.59	136.59
143.86	143.90	143.90
150.83	150.87	150.88
157.47	157.51	157.51

Relative errors on natural frequencies (%)	
SOBI	**NoHeJAD**
-0.01	-0.01
0.13	0.13
-0.05	-0.05
-0.10	-0.11
-0.01	0.00
-0.20	-0.22
-0.08	-0.05
-0.16	-0.17
-0.08	-0.09
0.03	-0.01
-0.03	-0.03
-0.03	-0.03
-0.03	-0.03
-0.03	-0.03
-0.03	-0.03
Averaged (absolute) relative errors (%)	
0.07	0.07

Table 2: Estimated damping ratios

Damping ratios ζ_n (%)		
True	**SOBI**	**NoHeJAD**
1.00	1.18	1.18
1.94	1.92	1.93
2.74	2.67	2.68
2.91	2.96	2.96
3.34	3.42	3.40
3.35	3.16	3.17
3.72	3.75	3.77
3.86	3.97	3.97
4.30	4.09	4.10
4.59	4.58	4.60
2.61	2.62	2.62
2.46	2.45	2.45
2.33	2.32	2.32
2.22	2.21	2.21
2.12	2.13	2.13

Relative errors on damping ratios (%)	
SOBI	**NoHeJAD**
-17.53	-17.80
0.70	0.42
2.23	2.13
-1.71	-1.64
-2.58	-1.87
5.57	5.19
-0.86	-1.36
-2.78	-2.85
4.79	4.58
0.31	-0.13
-0.37	-0.39
0.32	0.34
0.21	0.20
0.29	0.33
-0.17	-0.17
Averaged (absolute) relative errors (%)	
2.69	2.63

Table 3: MACs on real and imaginary parts of estimated mode shapes

MAC on real parts (%)	
SOBI	NoHeJAD
100.00	100.00
100.00	100.00
100.00	99.98
99.99	99.95
94.09	90.99
92.04	99.63
99.96	99.96
99.98	99.81
99.92	99.99
99.86	99.97
100.00	99.87
100.00	99.99
99.98	99.96
99.96	99.85
99.97	99.96
Averaged MACs	
99.05	99.33

MAC on imaginary parts (%)	
SOBI	NoHeJAD
NA	42.22
NA	27.18
NA	17.14
NA	53.74
NA	94.52
NA	84.43
NA	55.34
NA	69.93
NA	42.98
NA	25.44
NA	99.99
NA	99.37
NA	53.57
NA	99.86
NA	100.00
Averaged MACs	
NA	64.38

4. Conclusions

A new algorithm is suggested in this paper that enables utilization of second order blind source separation (SOBSS) techniques for operational modal analysis. This algorithm is coined Non-Hermitian Joint Approximate Diagonalization algorithm and as the name suggests involves joint diagonalization of correlation matrices at various time lags such that the factorization is Non-Hermitian. The Non-Hermitian factorization step makes this algorithm different from usual form of SOBSS algorithms, in the process enabling one to estimate systems having complex mode shapes and heavily damped modes; issues which have limited use of SOBSS algorithms in OMA domain. It is further shown in this work, that the Hankel matrix based general form of NoHeJAD is also capable of dealing with situations where the number of sensors available are less than the number of modes of interest; yet another issue where SOBSS algorithms have been found wanting. The algorithm has good convergence and has been shown to outperform Second Order Blind Identification (SOBI) algorithm. These encouraging results serve as motivation to apply and test NoHeJAD on practical real world OMA problem.

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