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Using the unified matrix polynomial approach (UMPA) for the development of the stochastic subspace identification (SSI) algorithm

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Abstract

The unified matrix polynomial approach (UMPA) was developed in order to understand and derive various experimental modal analysis algorithms (which have been developed in isolation) using a common mathematical formulation. Various commercially available algorithms – such as the polyreference time domain, least squares complex exponential, and eigensystem realization algorithm etc. – can be explained using UMPA methodology, which makes it easier to understand both the advantages and limitations of such algorithms. In view of this fundamental characteristic of the UMPA, this paper aims at using the approach to understand, explain and develop the stochastic subspace identification (SSI) algorithm - a popular time domain operational modal analysis (OMA) algorithm. The roots of SSI algorithm lie in the identification of linear dynamic systems, traditionally a communications and controls engineering area. By means of the UMPA, the SSI algorithm's similarity to a high order time domain OMA algorithm can be shown. It can also be shown that state transition matrices identified using the SSI algorithm and UMPA formulation are related to each other through a similarity transformation, thus characterizing the same system.

Keywords

Operational modal analysis, polynomial approach, state transition matrix, stochastic subspace identification, unified matrix polynomial approach

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1. Introduction

Modal parameter estimation is an important step in experimental modal analysis (EMA), the goal of which is to identify modal parameters of the structure - i.e. the modal frequency, damping and mode shape that characterize it. Over the years, modal parameter estimation has been an area of considerable research, and several parameter estimation algorithms which can assist in executing this task are now available. Historically, most of these algorithms have been developed independently to each other, making it difficult to understand the similarities they share. The unified matrix polynomial approach (UMPA) (Allemang et al., 1994; Allemang and Brown, 1998; Allemang and Phillips, 2004) is a mathematical concept that places a variety of modal parameter estimation algorithms within a consistent framework. Thus, the UMPA makes it easier to understand and compare these algorithms and evaluate their advantages and limitations. Furthermore, the UMPA aids in formulating these algorithms using a common mathematical framework.

The last 15 years or so have seen the emergence of a new methodology that aims at identifying dynamic characteristics of a structure solely on the basis of

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measured responses; this is in contrast to the traditional modal analysis, which requires knowledge of both input forces and output responses (Maia and Silva, 1997; Ewins, 2000). This methodology is referred to as output-only modal analysis or, more commonly, operational modal analysis (OMA) (Zhang et al., 2005). Due to its usefulness in understanding and developing modal parameter estimation algorithms, the concept of the UMPA was recently extended to OMA in Chauhan et al. (2007). This work has come to illustrate how fundamental data (power spectra and correlation functions) should be used by means of the UMPA model to develop various time, frequency and spatial domain OMA algorithms.

The work presented in this paper aims at using higher order time domain UMPA formulation for OMA to derive the stochastic subspace identification (SSI) algorithm.

The SSI is a popular time domain OMA algorithm based on a parametric state-space model (Van Overschee and De Moor, 1996; Peeters and De Roeck, 2001; Brincker and Andersen, 2006). It is worth noting that the use of state-space models for modal parameter estimation is not new. In fact, such models have also been used in traditional modal analysis. The eigensystem realization algorithm (ERA) (Juang and Pappa, 1985; Longman and Juang, 1989) and Ibrahim time domain (ITD) algorithm (Ibrahim and Mikulcik, 1977; Pappa, 1982) are examples of such algorithms in the EMA domain which utilize state-space models. Equivalent UMPA formulation of these algorithms can be found in Allemang and Brown (1998) and Allemang and Phillips (2004).

This paper has three main objectives:

a. To derive the SSI algorithm by using UMPA formulation for OMA in the time domain.
b. To explain the relationship between state transition matrix obtained using conventional SSI formulation and that obtained using UMPA formulation.
c. To explain the fundamental similarities (and differences) between higher order and state-space formulations for modal parameter estimation algorithm.

The paper will first introduce the UMPA model in the OMA domain and then the covariance based SSI (SSI-COV) algorithm. This will be followed by an explanation of the theoretical relation between higher order differential equations and corresponding state equations. Based on this knowledge, the UMPA model will be modified and utilized for developing the SSI-COV algorithm. Finally, theoretical concepts will be illustrated by means of a simple five degrees-of-freedom (d.f.) system.

2. Theoretical background

2.1. Revisiting the unified matrix polynomial approach (UMPA) model

In order to understand the UMPA, a matrix equation of motion for a general multi d.f. system is considered

$$\mathbf{M}\ddot{\mathbf{x}}(t) + \mathbf{C}\dot{\mathbf{x}}(t) + \mathbf{K}\mathbf{x}(t) = \mathbf{f}(t) \tag{1}$$

where:

$\mathbf{M}$ is the mass matrix,
$\mathbf{C}$ is the damping matrix,
$\mathbf{K}$ is the stiffness matrix,
$\mathbf{x}(t)$ is the response vector
$\mathbf{f}(t)$ is the force vector.

This is a second order differential equation that can be solved either in the time, frequency or Laplace domain. The characteristic equation for the system can be obtained by Laplace transforming equation (1). Thus

$$\left[\mathbf{M}s^2 + \mathbf{C}s + \mathbf{K}\right]\mathbf{x}(s) = \mathbf{f}(s) \tag{2}$$

Under free vibration conditions, the characteristic equation becomes

$$\left|\mathbf{M}s^2 + \mathbf{C}s + \mathbf{K}\right| = 0 \tag{3}$$

The partitioned form of the above equation can be written as follows

$$\left| \begin{bmatrix} \mathbf{M}_{11} & \mathbf{M}_{12} & \cdots & \mathbf{M}_{1m} \\ \mathbf{M}_{21} & \mathbf{M}_{22} & \cdots & \mathbf{M}_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{M}_{m1} & \mathbf{M}_{m2} & \cdots & \mathbf{M}_{mm} \end{bmatrix} s^2 + \begin{bmatrix} \mathbf{C}_{11} & \mathbf{C}_{12} & \cdots & \mathbf{C}_{1m} \\ \mathbf{C}_{21} & \mathbf{C}_{22} & \cdots & \mathbf{C}_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{C}_{m1} & \mathbf{C}_{m2} & \cdots & \mathbf{C}_{mm} \end{bmatrix} s + \begin{bmatrix} \mathbf{K}_{11} & \mathbf{K}_{12} & \cdots & \mathbf{K}_{1m} \\ \mathbf{K}_{21} & \mathbf{K}_{22} & \cdots & \mathbf{K}_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{K}_{m1} & \mathbf{K}_{m2} & \cdots & \mathbf{K}_{mm} \end{bmatrix} \right| = \mathbf{0} \tag{4}$$

This second order model can be converted into a higher order model in order to handle spatial information being truncated to a size smaller than the number of eigenvalues in the measured data. Thus equation (4)

can be expanded to a higher order matrix polynomial and noted in a more generic form as

$$\boldsymbol{\alpha}_{\mathbf{2m}} s^{2m} + \boldsymbol{\alpha}_{\mathbf{2m-1}} s^{2m-1} + \cdots\cdots + \boldsymbol{\alpha}_{\mathbf{0}} = \mathbf{0} \tag{5}$$

Note that the size of $\boldsymbol{\alpha}$ is same as the size of the partioned sub-matrices, and that each $\boldsymbol{\alpha}$ matrix involves a matrix product and summation of several $\mathbf{M_{ij}}$, $\mathbf{C_{ij}}$ and $\mathbf{K_{ij}}$ sub-matrices. The higher order equation, equation (5), has the same eigenvalues as the original second order differential equation, equation (1). In other words, irrespective of the representation, the obtained dynamic characteristics of the system will remain the same.

The general equation of motion in equation (1) can also be expressed as higher order differential equations in frequency and Laplace domains using matrix polynomial formulations. Generally these equations are represented in terms of frequency response functions (FRF) $\mathbf{H}(\omega)$ and

Transfer functions $\mathbf{H}(s)$ as

Frequency domain

$$\begin{aligned} &\left[\boldsymbol{\alpha}_{\mathbf{m}}(j\omega)^m + \boldsymbol{\alpha}_{\mathbf{m-1}}(j\omega)^{m-1} + \cdots\cdots + \boldsymbol{\alpha}_{\mathbf{0}}\right]\mathbf{x}(\omega) \\ &\quad = \left[\boldsymbol{\beta}_{\mathbf{n}}(j\omega)^n + \boldsymbol{\beta}_{\mathbf{n-1}}(j\omega)^{n-1} + \cdots\cdots + \boldsymbol{\beta}_{\mathbf{0}}\right]\mathbf{f}(\omega) \end{aligned}$$

or

$$\begin{aligned} &\left[\boldsymbol{\alpha}_{\mathbf{m}}(j\omega)^m + \boldsymbol{\alpha}_{\mathbf{m-1}}(j\omega)^{m-1} + \cdots\cdots + \boldsymbol{\alpha}_{\mathbf{0}}\right]\mathbf{H}(\omega) \\ &\quad = \left[\boldsymbol{\beta}_{\mathbf{n}}(j\omega)^n + \boldsymbol{\beta}_{\mathbf{n-1}}(j\omega)^{n-1} + \cdots\cdots + \boldsymbol{\beta}_{\mathbf{0}}\right]\mathbf{I} \end{aligned} \tag{6}$$

Laplace domain

$$\begin{aligned} &\left[\boldsymbol{\alpha}_{\mathbf{m}}(s)^m + \boldsymbol{\alpha}_{\mathbf{m-1}}(s)^{m-1} + \cdots\cdots + \boldsymbol{\alpha}_{\mathbf{0}}\right]\mathbf{x}(s) \\ &\quad = \left[\boldsymbol{\beta}_{\mathbf{n}}(s)^n + \boldsymbol{\beta}_{\mathbf{n-1}}(s)^{n-1} + \cdots\cdots + \boldsymbol{\beta}_{\mathbf{0}}\right]\mathbf{f}(s) \end{aligned}$$

or

$$\begin{aligned} &\left[\boldsymbol{\alpha}_{\mathbf{m}}(s)^m + \boldsymbol{\alpha}_{\mathbf{m-1}}(s)^{m-1} + \cdots\cdots + \boldsymbol{\alpha}_{\mathbf{0}}\right]\mathbf{H}(s) \\ &\quad = \left[\boldsymbol{\beta}_{\mathbf{n}}(s)^n + \boldsymbol{\beta}_{\mathbf{n-1}}(s)^{n-1} + \cdots\cdots + \boldsymbol{\beta}_{\mathbf{0}}\right]\mathbf{I} \end{aligned} \tag{7}$$

In the above equations, m usually represents the model order (theoretically related to d.f. of the system) and is the order of denominator polynomial whereas n represents the order of numerator polynomial (related to the nature of input excitation).

The equivalent relation in the time domain is given as

Time domain

$$\begin{aligned} &\boldsymbol{\alpha}_{\mathbf{m}} \frac{\mathrm{d}^m \mathbf{x}(t)}{\mathrm{d}t^m} + \boldsymbol{\alpha}_{\mathbf{m-1}} \frac{\mathrm{d}^{m-1} \mathbf{x}(t)}{\mathrm{d}t^{m-1}} + - - - - - - + \boldsymbol{\alpha}_{\mathbf{0}} \mathbf{x}(t) \\ &\quad = \boldsymbol{\beta}_{\mathbf{n}} \frac{\mathrm{d}^n \mathbf{f}(t)}{\mathrm{d}t^n} + \boldsymbol{\beta}_{\mathbf{n-1}} \frac{\mathrm{d}^{n-1} \mathbf{f}(t)}{\mathrm{d}t^{n-1}} + - - - - - - - + \boldsymbol{\beta}_{\mathbf{0}} \mathbf{f}(t) \end{aligned} \tag{8a}$$

which can be written in the discrete time domain in terms of the impulse response function (IRF), $h(t)$, for sampled data as

$$\boldsymbol{\alpha}_{\mathbf{m}} \mathbf{h}(t_m) + \boldsymbol{\alpha}_{\mathbf{m-1}} \mathbf{h}(t_{m-1}) + \cdots\cdots + \boldsymbol{\alpha}_{\mathbf{0}} \mathbf{h}(t_0) = \mathbf{0} \tag{8b}$$

where $\mathbf{h}$ represents IRF matrices at various time lags and $\boldsymbol{\alpha}$ represents coefficient matrices.

To understand the model further, the polynomial model for FRF - as described in equation (6) - is considered. For a particular FRF $H_{pq}(\omega)$, where p and q are response and excitation d.f. respectively, equation (6) can be written as

$$H_{pq}(\omega_i) = \frac{X_p(\omega_i)}{F_q(\omega_i)} = \frac{\beta_n(j\omega)^n + \beta_{n-1}(j\omega)^{n-1} + - - - - - + \beta_0(j\omega)^0}{\alpha_m(j\omega)^m + \alpha_{m-1}(j\omega)^{m-1} + - - - - - + \alpha_0(j\omega)^0} \tag{9}$$

This can be rewritten as

$$H_{pq}(\omega_i) = \frac{X_p(\omega_i)}{F_q(\omega_i)} = \frac{\sum_{k=0}^{n} \beta_k (j\omega)^k}{\sum_{k=0}^{m} \alpha_k (j\omega)^k} \tag{10}$$

Or, for a general multiple input, multiple output case

$$\sum_{k=0}^{m} (j\omega)^k \boldsymbol{\alpha}_{\mathbf{k}} \mathbf{H}(\omega) = \sum_{k=0}^{n} (j\omega)^k \boldsymbol{\beta}_{\mathbf{k}} \tag{11}$$

Note that the size of the coefficient matrices is normally N_i X N_i for $\boldsymbol{\alpha}_{\mathbf{k}}$ and N_i X N_o for $\boldsymbol{\beta}_{\mathbf{k}}$ where N_i and N_o are the number of input and output d.f. respectively. (In other words, N_i is the number of d.f. where force is applied to the structure and N_o is the number of d.f. where the response to these forces is measured.)

As shown previously in equation (8b), the equivalent model of equation (11) in the time domain is given in terms of IRFs $\mathbf{h}(t)$. This can be written in a more concise form as shown below (with $\mathbf{0}$ being a null matrix).

$$\sum_{k=0}^{m} \boldsymbol{\alpha}_{\mathbf{k}} \mathbf{h}(t) = \mathbf{0} \tag{12}$$

The general matrix polynomial model concept, as shown in equations (11) and (12), recognizes that both time and frequency domain models generate functionally similar matrix polynomial models. This model, which describes both domains, is thus termed the UMPA. Further details of the UMPA and its effectiveness in understanding and developing various modal parameter estimation algorithms can be found in Allemang et al. (1994), Allemang and Brown (1998) and Allemang and Phillips (2004).

To realize the obvious potential that the UMPA has with regard to overall understanding of the modal

parameter estimation procedure, it is extended to OMA in Chauhan et al. (2007), where it is shown how fundamental data (correlation functions and power spectra, instead of IRF and FRF), should be used within the UMPA framework for developing various OMA algorithms. The UMPA equivalent equations, corresponding to equations (11) and (12), in the OMA domain, are presented in terms of positive power spectra G^{+}_{XX} (Cauberghe, 2004) in the frequency domain (for any given frequency ω_i) and correlation function R_{XX} in the time domain (for any given time lag t_i) as [7]

$$\sum_{k=0}^{m} (j\omega)^k \boldsymbol{\alpha}_\mathbf{k} \mathbf{G}^{+}_{\mathbf{XX}}(\omega_i) = \sum_{k=0}^{n} (j\omega)^k \boldsymbol{\beta}_\mathbf{k} \tag{13}$$

$$\sum_{k=0}^{m} \boldsymbol{\alpha}_\mathbf{k} \mathbf{R}_{\mathbf{XX}}(t_{i+k}) = \mathbf{0} \tag{14}$$

It is worth noting that the use of positive power spectra is necessary for accurate modal parameter estimation within the OMA domain. This can be explained by means of the partial fraction model of power spectra, which reveals that the power spectra contains information about the system modes twice - both as actual system modes as well as corresponding modes with negative damping. In other words, the power spectra formulation contains the same system-related information twice, resulting in complications in the parameter estimation procedure. This issue of negatively damped modes is avoided by using positive power spectra instead of power spectra. For a further explanation related to the application of positive power spectra (Chauhan et al., 2007; Cauberghe, 2004).

Unlike EMA, in equations (13) and (14) N_i is the number of outputs being considered as references and thus it is not exactly inputs as is the case with equations (11) and (12). It should be further noted that only positive lags of the correlation function are used for the above formulation.

An important aspect of the UMPA formulation (both in EMA and OMA forms), as shown in equations (11) to (14), is to understand the relationship between the number of roots (or modal parameters) of the system and the model order m and size of the coefficient matrices $\boldsymbol{\alpha}$. The total number of roots is equal to either mN_i or mN_o depending on whether the equations are formed on the basis of the number of references (or inputs in case of EMA) or number of outputs.

2.2. The covariance based stochastic subspace identification (SSI-COV) algorithm

The SSI algorithm is a well-known and commonly used OMA algorithm that utilizes a state-space formulation based approach to modal parameter identification using output-only data. The SSI algorithm stems from usage within the area of controls engineering, where state-space models have been used for designing optimal control systems and defining the controllability and observability of linear systems. As mentioned earlier, state-space models have also been utilized for system identification purposes and have been known to the modal analysis community in both the EMA and OMA domain (the ERA, ITD and SSI are some of the algorithms based on the state-space model).

There are several good resources that describe the SSI algorithm in detail (Peeters and De Roeck, 2001; Van Overschee and De Moor, 1996; Brincker and Andersen, 2006). This Section presents the algorithm briefly, laying down the fundamentals for its subsequent development and understanding using the UMPA.

The SSI-COV algorithm is based on the formulation of the stochastic state-space model using measured output responses. In the discrete time domain, this model is represented by the following equations (not taking into account the process and measurement noise)

$$\begin{aligned} \mathbf{y_{k+1}} &= \mathbf{A}\mathbf{y_k} \\ \mathbf{y_k} &= \mathbf{C}\mathbf{x_k} \end{aligned} \tag{15}$$

where $\mathbf{x}$ is the vector of measured responses, $\mathbf{y}$ is the vector of state variables, matrix $\mathbf{A}$ is the state transition matrix (which contains all necessary information about the dynamic characteristics of the system), and $\mathbf{C}$ is the output matrix. The covariance matrix of output response signals $\mathbf{x}$ can now be expressed as

$$\mathbf{R_i} = E\left[\mathbf{x_k}\mathbf{x}_{\mathbf{k-i}}^T\right] = \mathbf{C}\mathbf{A}^{i-1}\mathbf{G} \tag{16}$$

where $\mathbf{G} = E\left[\mathbf{y_{k+1}}\ \mathbf{x}_\mathbf{k}^T\right]$ is the state-output covariance matrix. As mentioned earlier, the state transition matrix contains necessary information regarding system dynamics and this information is described by the eigenvalues and eigenvectors of the state transition matrix $\mathbf{A}$, eigenvalues λ_r being the diagonal values of $\boldsymbol{\Lambda}$ and eigenvectors $\boldsymbol{\Psi}_\mathbf{r}$ being given by $\boldsymbol{\Psi}_\mathbf{r} = \mathbf{C}\boldsymbol{\varphi}_\mathbf{r}$, where $\boldsymbol{\varphi}_\mathbf{r}$ is the r^{th} column of $\boldsymbol{\Phi}$ corresponding to the r^{th} eigenvalue λ_r obtained after eigenvalue decomposition of the state transition matrix $\mathbf{A}$.

$$\mathbf{A} = \boldsymbol{\Phi}\boldsymbol{\Lambda}\boldsymbol{\Phi}^{-1} \tag{17}$$

The procedure of estimating a system's modal parameters from measured output responses in an output-only framework requires an estimation of the state transition matrix $\mathbf{A}$ and output matrix $\mathbf{C}$. This is typically achieved by the decomposition of the block Hankel

matrix of output correlation functions into observability and controllability matrices based on the following relation

$$\mathbf{H_{p,q}} = E\left[\mathbf{x_k x_{k-1}^T}\right] = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \mathbf{CA}^2 \\ \vdots \\ \mathbf{CA}^{p-1} \end{bmatrix} \times \begin{bmatrix} \mathbf{G} & \mathbf{AG} & \mathbf{A}^2\mathbf{G} & \cdots \mathbf{A}^{q-1}\mathbf{G} \end{bmatrix} = \mathbf{O_p C_q} \tag{18}$$

Equation (19) shows the block Hankel matrix of correlation matrices $\mathbf{R_i}$, where i indicates the time lag.

$$\mathbf{H_{p,q}} = \begin{bmatrix} \mathbf{R_1} & \mathbf{R_2} & \cdots & \mathbf{R_q} \\ \mathbf{R_2} & \mathbf{R_3} & \cdots & \mathbf{R_{q+1}} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{R_p} & \mathbf{R_{p+1}} & \cdots & \mathbf{R_{p+q-1}} \end{bmatrix} \tag{19}$$

The block Toeplitz form can also be used, as shown in Peeters and De Roeck (1999), instead of the block Hankel matrix to achieve the same goal. The block Hankel matrix can be decomposed into observability and controllability matrices, $\mathbf{O_p}$ and $\mathbf{C_q}$, by performing singular value decomposition of $\mathbf{H_{p,q}}$

$$\begin{aligned} \mathbf{H_{p,q}} &= \mathbf{USV}^T \\ \mathbf{O_p} &= \mathbf{US}^{1/2} \\ \mathbf{C_q} &= \mathbf{S}^{1/2}\mathbf{V}^T \end{aligned} \tag{20}$$

The procedure for estimating the state transition matrix $\mathbf{A}$ after calculating the observability and controllability matrices is straightforward. The output matrix $\mathbf{C}$ is given by

$$\mathbf{C} = \text{First block row of } \mathbf{O_p} \tag{21}$$

and the state transition matrix is obtained using

$$\mathbf{A} = \mathbf{O}_{\mathbf{p-1}}^{T^+}\mathbf{O}_{\mathbf{p-1}}^{\uparrow} \tag{22}$$

where $\mathbf{O_{p-1}}$ is obtained by deleting the last block row of $\mathbf{O_p}$, $\mathbf{O}_{\mathbf{p-1}}^{\uparrow}$ is the upper-shifted matrix by one block row and $^+$ represents pseudo-inverse.

In the context of this work, it is important to understand that the state transition matrix $\mathbf{A}$, which contains information about dynamic characteristics of the system, can also be estimated using the block shifted Hankel matrix. This formulation is explained in terms of the block shifted Toeplitz matrix in Peeters and De Roeck (1999). The block shifted Hankel matrix can be expressed in terms of observability, controllability and state transition matrices as

$$\overleftarrow{\mathbf{H}}_{\mathbf{p,q}} = E\left[\mathbf{x_{k+1}\ x_{k-1}^T}\right] = \mathbf{O_p A C_q} \tag{23}$$

and thus $\mathbf{A}$ can be estimated as

$$\mathbf{A} = \mathbf{O_p^+}\ \overleftarrow{\mathbf{H}}_{\mathbf{p,q}}\ \mathbf{C_q^+} \tag{24}$$

The above formulation of the SSI is well-known and the state transition matrix $\mathbf{A}$ is typically estimated using either of the two equations; equation (22) or equation (24).

However, from a modal parameter estimation perspective, since the ultimate aim is to identify modal parameters, it is mathematically possible to estimate the state transition matrix $\mathbf{A}$ directly from the Hankel matrix of correlations without needing to first obtain the observability and controllability matrices. This is explained in the following discussion.

From equation (18), it is easy to understand that decomposition of $\mathbf{H_{p,q}}$ into $\mathbf{O_p C_q}$ is not unique. Substituting $\mathbf{O_p}$ as the identity matrix $\mathbf{I}$ in equation (18) results in $\mathbf{C_q}$ being equivalent to $\mathbf{H_{p,q}}$; i.e. $\mathbf{C_q} = \mathbf{H_{p,q}}$. The block shifted Hankel matrix of equation (23) can now be written as

$$\overleftarrow{\mathbf{H}}_{\mathbf{p,q}} = E\left[\mathbf{x_{k+1}\ x_{k-1}^T}\right] = \mathbf{O_p A C_q} = \mathbf{I A H_{p,q}}$$

by substituting $\mathbf{O_p} = \mathbf{I}$ and $\mathbf{C_q} = \mathbf{H_{p,q}}$. In such a case, the state transition matrix $\mathbf{A}$ can be expressed as

$$\mathbf{A_2} = \mathbf{A} = \mathbf{H}_{\mathbf{p,q}}^{+}\overleftarrow{\mathbf{H}}_{\mathbf{p,q}} \tag{25}$$

Note that $\mathbf{A_2}$ represents an estimation of the state transition matrix using the Hankel matrix directly, as shown in equation (25). The importance of this result will be clear later, when the UMPA is utilized for deriving the SSI algorithm and it is shown that $\mathbf{A_2}$ is the companion matrix (Horn and Charles, 1985). For the sake of simplicity, $\mathbf{A_2}$ will continue to be referred to as the state transition matrix - albeit after a direct estimation from the Hankel matrix.

The fact that the state transition matrix $\mathbf{A}$ (as estimated using the observability matrix) and $\mathbf{A_2}$ (as estimated from the Hankel matrix) are related by a similarity transformation means that they will have the same eigenvalues and eigenvectors. At this point then it is good to review the concept of similarity transformation and properties of similar matrices.

2.2.1. Similarity transformation and properties of similar matrices

It is a well-known fact that two similar matrices (those related through a similarity transformation) have the same eigenstructure, thus having the same eigenvalues and eigenvectors. This also applies to characteristic equations (and transfer functions) (Kuo, 1995). It is now shown how matrices $\mathbf{A}$ and $\mathbf{A_2}$, estimated using the observability matrix and directly from the Hankel matrix respectively, are related to each other through a similarity transformation, thus yielding essentially the same system characteristics (eigenvalues and eigenvectors).

Consider equation (23)

$$\overleftarrow{\mathbf{H}}_{\mathbf{p,q}} = \mathbf{O_p A C_q}$$

In equation (23), $\overleftarrow{\mathbf{H}}_{\mathbf{p,q}}$ can be substituted in terms of $\mathbf{A_2}$ and $\mathbf{H_{p,q}}$ using equation (24) to form a relationship between $\mathbf{A}$ and $\mathbf{A_2}$

$$\mathbf{A_2 H_{p,q}} = \mathbf{O_p A C_q}$$

This can be further expanded through singular value decomposition based definitions of $\mathbf{O_p}$, $\mathbf{C_q}$ and $\mathbf{H_{p,q}}$, as described in equation (20), in the following manner

$$\begin{aligned} \mathbf{A_2 U S V}^T &= \mathbf{U S}^{1/2} \mathbf{A S}^{1/2} \mathbf{V}^T \\ \mathbf{A_2 U S}^{1/2} \mathbf{S}^{1/2} \mathbf{V}^T &= \mathbf{U S}^{1/2} \mathbf{A S}^{1/2} \mathbf{V}^T \\ \mathbf{A_2} &= \mathbf{U S}^{1/2} \mathbf{A S}^{-1/2} \mathbf{U}^T \end{aligned} \tag{26}$$

This shows that the matrix $\mathbf{P} = \mathbf{US}^{1/2}$, relating $\mathbf{A}$ and $\mathbf{A_2}$, is a similarity transformation - thus implying that $\mathbf{A}$ and $\mathbf{A_2}$ have the same eigenvalues and eigenvectors or, in other words, that both matrices characterize dynamics of the same system.

Along similar lines, it can also be shown that similarity transformation preserves the characteristic equation of $\mathbf{A}$ - i.e. the transformed system $\mathbf{A_2}$ also has the same characteristic equation as $\mathbf{A}$. The characteristic equation for the transformed system can be written as $|s\mathbf{I} - \mathbf{A_2}| = \mathbf{0}$, which can be expressed in terms of similarity transformation matrix $\mathbf{P}$ and $\mathbf{A}$ as

$$\begin{aligned} &|s\mathbf{I} - \mathbf{A_2}| \\ &= \left|s\mathbf{I} - \mathbf{PAP}^{-1}\right| \\ &= \left|s\mathbf{PP}^{-1} - \mathbf{PAP}^{-1}\right| \\ &= |\mathbf{P}|\,|s\mathbf{I} - \mathbf{A}|\,\left|\mathbf{P}^{-1}\right| \\ &= |s\mathbf{I} - \mathbf{A}| \end{aligned} \tag{27}$$

Since the characteristic equation is preserved, the two matrices will have the same eigenvalues and eigenvectors; in other words, $\mathbf{A}$ and $\mathbf{A_2}$ represent the same dynamic system and will thus yield the same modal parameters.

3. Deriving the covariance based stochastic subspace identification (SSI-COV) algorithm using the unified matrix polynomial approach (UMPA)

3.1. Relationship between higher order differential equations and state equations

Having understood the theoretical aspects of the SSI algorithm, the focus of this section is on understanding its relationship with higher order modal parameter estimation algorithms such as the polyreference time domain (PTD) (Vold and Rocklin, 1982; Vold et al., 1982) algorithm. The key to understanding this connection lies in the basic concept of representing a higher order differential equation in terms of equivalent state equations.

In general, a differential equation of m^{th} order can be decomposed into m first-order differential equations. Consider such an m^{th} order differential equation as the one underneath in equation (28), analogous to the left hand side of equation (8a) and representing a general m d.f., MIMO system.

$$\boldsymbol{\alpha}_m \frac{\mathrm{d}^m \mathbf{x}(t)}{\mathrm{d}t^m} + \boldsymbol{\alpha}_{m-1} \frac{\mathrm{d}^{m-1} \mathbf{x}(t)}{\mathrm{d}t^{m-1}} + - - - - - - + \boldsymbol{\alpha}_0 \mathbf{x}(t) = 0 \tag{28}$$

The above equation can be rearranged as follows (note that division by $\boldsymbol{\alpha}_m$ is incorporated in the definition of $\bar{\boldsymbol{\alpha}}$)

$$\frac{\mathrm{d}^m \mathbf{x}(t)}{\mathrm{d}t^m} = -\bar{\boldsymbol{\alpha}}_{m-1} \frac{\mathrm{d}^{m-1} \mathbf{x}(t)}{\mathrm{d}t^{m-1}} - \bar{\boldsymbol{\alpha}}_{m-2} \frac{\mathrm{d}^{m-2} \mathbf{x}(t)}{\mathrm{d}t^{m-2}} - \cdots\cdots - \bar{\boldsymbol{\alpha}}_0 \mathbf{x}(t) \tag{29}$$

To represent the above equation in its equivalent state-space form, a set of variables can be defined, such that

$$\begin{aligned} \mathbf{y}_1(t) &= \mathbf{x}(t) \\ \mathbf{y}_2(t) &= \frac{\mathrm{d}\mathbf{x}(t)}{\mathrm{d}t} \\ &\vdots \\ &\vdots \\ \mathbf{y}_m(t) &= \frac{\mathrm{d}^{m-1} \mathbf{x}(t)}{\mathrm{d}t^{m-1}} \end{aligned} \tag{30}$$

The variables $\mathbf{y}_1, \mathbf{y}_2 \ldots\ldots\ldots \mathbf{y}_m$ are called state variables. These can be utilized to form a set of m first order differential equations which are known as the state equations.

The state equations are written as

$$\begin{aligned}
\frac{\mathrm{d}\mathbf{y}_1(t)}{\mathrm{d}t} &= \mathbf{y}_2(t) \\
\frac{\mathrm{d}\mathbf{y}_2(t)}{\mathrm{d}t} &= \mathbf{y}_3(t) \\
&\vdots \\
&\vdots \\
\frac{\mathrm{d}\mathbf{y}_m(t)}{\mathrm{d}t} &= -\bar{\boldsymbol{\alpha}}_{m-1}\mathbf{y}_m(t) - \bar{\boldsymbol{\alpha}}_{m-2}\mathbf{y}_{m-1}(t) - \cdots\cdots - \bar{\boldsymbol{\alpha}}_0\mathbf{y}_1(t)
\end{aligned} \tag{31}$$

It is important to note that although equations (31) and (28) appear to be very different representations, in essence they represent the same system (characterized by the same scalar polynomial coefficients).

Typically, equation (31) is written in the condensed form

$$\frac{\mathrm{d}\mathbf{Y}(t)}{\mathrm{d}t} = \mathbf{A}\mathbf{Y}(t) \tag{32}$$

where the state vector $\mathbf{Y}(t)$ and matrix $\mathbf{A}$ are defined as

$$\mathbf{Y}(t) = \begin{bmatrix} \mathbf{y}_1(t) \\ \mathbf{y}_2(t) \\ \vdots \\ \mathbf{y}_m(t) \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -\bar{\boldsymbol{\alpha}}_0 & -\bar{\boldsymbol{\alpha}}_1 & -\bar{\boldsymbol{\alpha}}_2 & \cdots & -\bar{\boldsymbol{\alpha}}_{m-1} \end{bmatrix} \tag{33}$$

It is clear that matrix $\mathbf{A}$ is the companion matrix of the high order differential equation equation (28), its eigenvalues being roots of the equation. The formulation of the companion matrix using polynomial coefficients is a standard way of finding the roots of a polynomial equation.

This discussion forms the basis of an important inference with regards to understanding the relationship between a typical higher order modal parameter estimation algorithm (e.g the PTD algorithm) and a state-space approach-based algorithm (e.g the SSI algorithm). A higher order algorithm is used to estimate polynomial coefficients which are then assembled in a companion matrix form; eigenvalue decomposition of this companion matrix yields the system poles. On the other hand, state-space algorithms estimate companion matrix directly. (Though typically they estimate state transition matrix, but state transition matrix is related to companion matrix through a similarity transformation. As mentioned in Section 2.2., SSI algorithm can be formulated directly from Hankel matrix of correlations to directly yield the companion matrix).

The knowledge gained from this section will be utilized in the upcoming section, where the UMPA equation for the higher order modal parameter estimation algorithm in the time domain, analogous to the PTD algorithm, is modified for the development of the SSI algorithm.

3.2. Using the unified matrix polynomial approach (UMPA) for deriving the covariance based stochastic subspace identification (SSI-COV) algorithm

Consider the time domain UMPA equation in the OMA domain, equation (14), with model order m

$$\sum_{k=0}^{m} \boldsymbol{\alpha}_\mathbf{k}\mathbf{R}_\mathbf{XX}(t_{i+k}) = \mathbf{0}$$

This equation can be expanded for a high order (using higher order coefficient normalization analogous to equation (29)) and can be written as underneath in equation (34), which represents the UMPA equation for the PTD, modified for the OMA domain (Chauhan et al., 2007).

$$\begin{aligned}
&\begin{bmatrix} \boldsymbol{\alpha}_0 & \boldsymbol{\alpha}_\mathbf{1} & \cdots \boldsymbol{\alpha}_{\mathbf{m-1}} \end{bmatrix}_{N_{ref} \times mN_{ref}} \begin{bmatrix} \mathbf{R}_{\mathbf{xx}}(t_{i+0}) \\ \mathbf{R}_{\mathbf{xx}}(t_{i+1}) \\ \vdots \\ \mathbf{R}_{\mathbf{xx}}(t_{i+m-1}) \end{bmatrix}_{mN_{ref} \times N_o} \\
&\quad = -\mathbf{R}_{\mathbf{xx}}(t_{i+m})_{N_{ref} \times N_o}
\end{aligned} \tag{34}$$

The number of system modes (N) that can be obtained using this equation is $N = m \times N_{ref,}$ where N_{ref} is the number of output responses considered as references.

Equation (34) can be repeated for other time lags and the matrix polynomial coefficients can be identified in a least square manner. This formulation is shown in

$$\begin{bmatrix} \mathbf{0} & \mathbf{I} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{I} \\ -\boldsymbol{\alpha}_{\mathbf{0}} & -\boldsymbol{\alpha}_{\mathbf{1}} & -\boldsymbol{\alpha}_{\mathbf{2}} & \cdots & -\boldsymbol{\alpha}_{\mathbf{m-1}} \end{bmatrix}_{mN_{ref} \times mN_{ref}}$$

$$\times \begin{bmatrix} \mathbf{R_{xx}}(t_{i+0}) \\ \mathbf{R_{xx}}(t_{i+1}) \\ \vdots \\ \mathbf{R_{xx}}(t_{i+m-2}) \\ \mathbf{R_{xx}}(t_{i+m-1}) \end{bmatrix}_{mN_{ref} \times N_o}$$

Comparing equations (39) and (25), and using the definition of the Hankel matrix from equation (19) (with $p = q$), it is clear that

$$\mathbf{H}_{\mathbf{p,q}} = \begin{bmatrix} \mathbf{R}_{\mathbf{xx}}^{\mathbf{i+0}} & \mathbf{R}_{\mathbf{xx}}^{\mathbf{i+1}} & \cdots & \mathbf{R}_{\mathbf{xx}}^{\mathbf{i+m-1}} \\ \mathbf{R}_{\mathbf{xx}}^{\mathbf{i+1}} & \mathbf{R}_{\mathbf{xx}}^{\mathbf{i+2}} & \cdots & \mathbf{R}_{\mathbf{xx}}^{\mathbf{i+m}} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{R}_{\mathbf{xx}}^{\mathbf{i+m-1}} & \mathbf{R}_{\mathbf{xx}}^{\mathbf{i+m}} & \cdots & \mathbf{R}_{\mathbf{xx}}^{\mathbf{i+2m-1}} \end{bmatrix}_{mN_{ref} \times mN_o}$$

$$\overleftarrow{\mathbf{H}}_{\mathbf{p,q}} = \begin{bmatrix} \mathbf{R}_{\mathbf{xx}}^{\mathbf{i+1}} & \mathbf{R}_{\mathbf{xx}}^{\mathbf{i+2}} & \cdots & \mathbf{R}_{\mathbf{xx}}^{\mathbf{i+m}} \\ \mathbf{R}_{\mathbf{xx}}^{\mathbf{i+2}} & \mathbf{R}_{\mathbf{xx}}^{\mathbf{i+3}} & \cdots & \mathbf{R}_{\mathbf{xx}}^{\mathbf{i+m+1}} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{R}_{\mathbf{xx}}^{\mathbf{i+m}} & \mathbf{R}_{\mathbf{xx}}^{\mathbf{i+m+1}} & \cdots & \mathbf{R}_{\mathbf{xx}}^{\mathbf{i+2m-1}} \end{bmatrix}_{mN_{ref} \times mN_o}$$

Thus, the state transition matrix $[A]_2$, obtained in equation (25) using the Hankel matrix-based SSI formulation, is essentially the companion matrix obtained using UMPA methodology. In this way, starting from the UMPA equivalent equation in the time domain for the OMA, it is possible to derive the SSI-COV algorithm. This process also helps in understanding the underlying relationship between a typical higher order identification algorithm and a state-space identification algorithm.

4. Numerical example

The purpose of this numerical example is not to comment on the accuracy of various algorithms but to facilitate understanding of their commonalities by demonstrating how they can be applied to numerically simulated data. It should be further emphasized that although minute differences between them exist, the algorithms are not separate ones but are different forms of the same algorithm. Thus it is expected that, due to the various signal processing steps involved, the estimated parameters will differ slightly from the theoretical solution and also from each other.

A simple five d.f. system is considered with the following mass (**M**), damping (**C**) and stiffness (**K**) matrices.

$$\mathbf{M} = \begin{bmatrix} 250 & 0 & 0 & 0 & 0 \\ 0 & 350 & 0 & 0 & 0 \\ 0 & 0 & 30 & 0 & 0 \\ 0 & 0 & 0 & 450 & 0 \\ 0 & 0 & 0 & 0 & 50 \end{bmatrix},$$

$$\mathbf{C} = \begin{bmatrix} 3250 & -250 & 0 & 0 & 0 \\ -250 & 450 & -200 & 0 & 0 \\ 0 & -200 & 320 & -120 & 0 \\ 0 & 0 & -120 & 190 & -70 \\ 0 & 0 & 0 & -70 & 270 \end{bmatrix}$$

$$\mathbf{K} = 1000 \times \begin{bmatrix} 9000 & -5000 & 0 & 0 & 0 \\ -5000 & 11000 & -6000 & 0 & 0 \\ 0 & -6000 & 12500 & -6500 & 0 \\ 0 & 0 & -6500 & 14500 & -8000 \\ 0 & 0 & 0 & 0 & 15000 \end{bmatrix}$$

The analytical modal frequency and damping of the system are shown in Table 1.

Response data is generated by exciting the system with a white random uncorrelated set of inputs at all d.f. The generated response is then processed using the Welch Periodogram approach (Stoica and Moses, 1997; Kay, 1988) in order to obtain auto and cross response power spectra. A 1024 block-size rectangular window is used along with a 66.67% overlap for data processing and a total of 300 averages are taken. Correlation

Table 1. Comparison of modal frequency and damping obtained using various algorithms

	Analytical		SSI-COV		UMPA-SSI/UMPA-PTD	
Mode number	Frequency (Hz)	Damping (%)	Frequency (Hz)	Damping (%)	Frequency (Hz)	Frequency (Hz)
1	12.526	1.149	12.541	1.217	12.535	1.185
2	22.083	1.059	22.102	1.086	22.097	1.066
3	34.864	2.172	34.913	2.206	34.891	2.145
4	88.524	0.487	88.506	0.506	88.526	0.479
5	104.78	0.847	104.72	0.821	104.76	0.780

SSI-COV: Covariance based stochastic subspace identification.
UMPA-SSI/UMPA-PTD: Unified matrix polynomial approach- stochastic subspace identification/unified matrix polynomial approach-polyreference time domain.

functions are then obtained by inverse Fourier transformation of averaged power spectra.

The SSI algorithm - the classic form of equation (22) and the UMPA form of equation (25) - and the UMPA based higher order algorithm (analogous to PTD equation (36)) are applied to correlation functions. To compare the performance of these algorithms consistently, the model order is kept constant at two and all five d.f. are considered as references. For the sake of simplicity, from here on the algorithms are referred to as SSI-COV (the classic form of the SSI algorithm), SSI-UMPA (SSI algorithm formulated using the UMPA) and UMPA-PTD (PTD algorithm formulated using the UMPA).

Table 1 and Figure 1 show a comparison of modal frequency and damping obtained using the three algorithms with the analytical solution. To compare the modal vector estimates, their values are presented in Table 2 along with the analytical modal vectors.

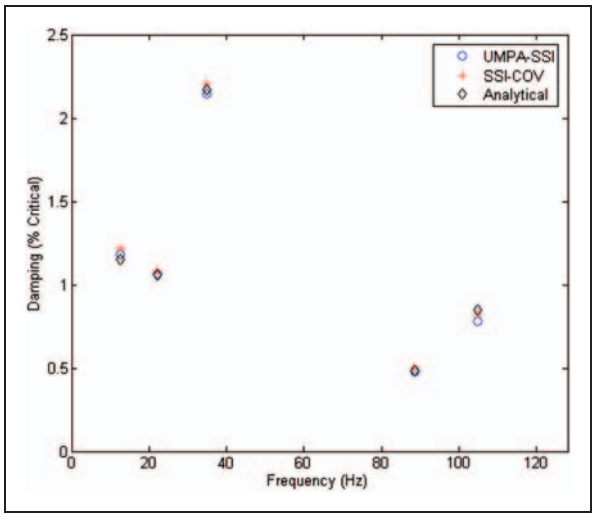

Figure 1. Comparison of analytical and estimated modal frequencies and damping.

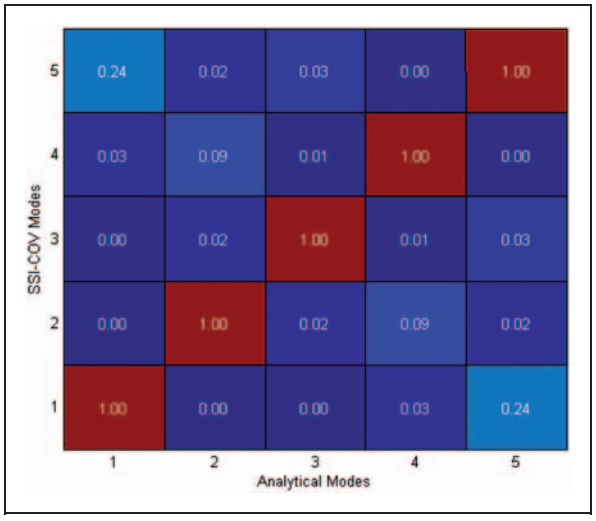

Figure 2. Cross modal assurance criterion (MAC) plot between analytical and covariance based stochastic subspace identification (SSI-COV) modal vectors.

Table 2. Comparison of modal vectors.*

		Degrees-of-freedom 1	Degrees-of-freedom 2	Degrees-of-freedom 3	Degrees-of-freedom 4	Degrees-of-freedom 5
Mode 1	A	1	$1.490 + 0.038i$	$1.360 + 0.046i$	$1.201 + 0.051i$	$0.654 + 0.027i$
	U	1	$1.491 - 0.038i$	$1.361 - 0.053i$	$1.204 - 0.050i$	$0.656 - 0.029i$
	S	1	$1.491 - 0.038i$	$1.361 - 0.053i$	$1.203 - 0.051i$	$0.655 - 0.029i$
Mode 2	A	1	$0.837 + 0.064i$	$-0.237 + 0.029i$	$-1.209 - 0.006i$	$-0.689 - 0.004i$
	U	1	$0.841 - 0.050i$	$-0.242 - 0.025i$	$-1.219 - 0.008i$	$-0.694 + 0.007i$
	S	1	$0.840 - 0.050i$	$-0.242 - 0.026i$	$-1.221 - 0.009i$	$-0.695 + 0.006i$
Mode 3	A	1	$-0.601 + 0.045i$	$-0.249 + 0.013i$	$0.132 - 0.016i$	$0.084 - 0.009i$
	U	1	$-0.608 - 0.029i$	$-0.252 + 0.007i$	$0.135 + 0.002i$	$0.084 - 0.004i$
	S	1	$-0.609 - 0.029i$	$-0.251 + 0.008i$	$0.134 + 0.002i$	$0.083 - 0.005i$
Mode 4	A	1	$-13.65 + 0.591i$	$220.4 - 11.86i$	$121.9 - 4.46i$	$-2080 + 68.45i$
	U	1	$-12.96 - 4.214i$	$205.5 + 75.45i$	$113.0 + 43.17i$	$-1917 - 758.5i$
	S	1	$-12.75 - 4.298i$	$201.7 + 76.99i$	$111.0 + 44.13i$	$-1881 - 75.01i$
Mode 5	A	1	$-19.86 + 0.717i$	$464.7 - 19.56i$	$-17.66 + 0.872i$	$21.14 - 1.517i$
	U	1	$-19.57 - 1.312i$	$455.5 + 47.77i$	$-17.38 - 1.306i$	$20.80 + 1.050i$
	S	1	$-19.29 - 1.527i$	$448.1 + 53.18i$	$-17.14 - 1.496i$	$20.56 + 1.301i$

*A- Analytical, U – UMPA-SSI (and UMPA-PTD), S – SSI-COV.

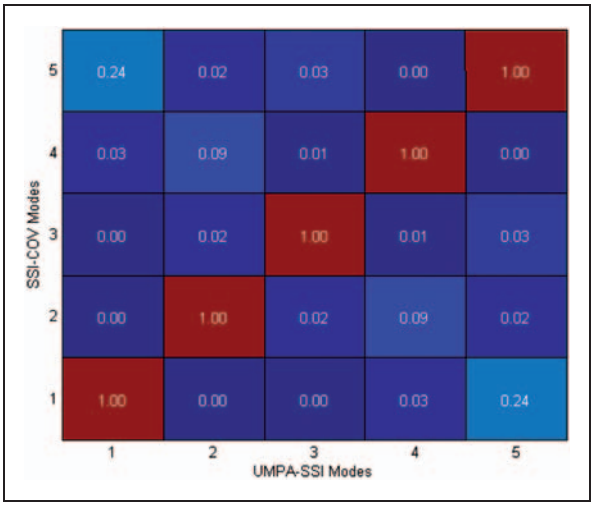

Figure 3. Cross modal assurance criterion (MAC) plot between unified matrix polynomial approach stochastic subspace identification (UMPA-SSI) and covariance based stochastic subspace identification (SSI-COV) modal vectors.

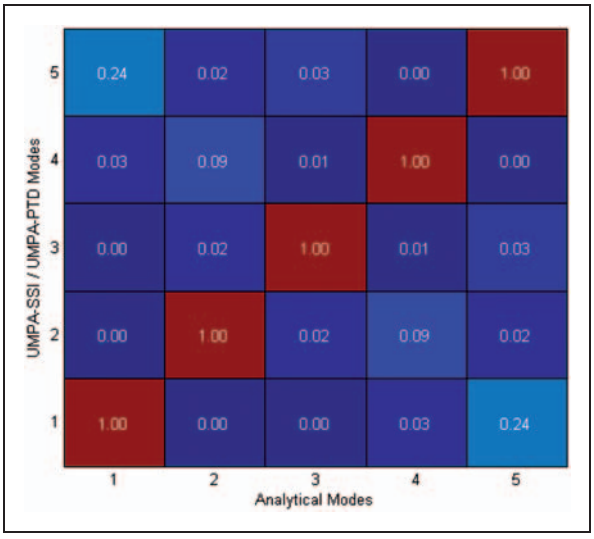

Figure 4. Cross modal assurance criterion (MAC) plot between analytical and unified matrix polynomial approach- stochastic subspace identification/unified matrix polynomial approach-polyreference time domain (UMPA-SSI/UMPA-PTD) modal vectors.

It should be noted that in Table 2 the vectors for each modes are normalized with respect to the first d.f. Additionally, for various comparisons, the modal assurance criterion (MAC) (Allemang, 2003) is calculated between the analytical and the estimated modal vectors and the MAC plots are shown in Figures 2 to 4.

Theoretically, various algorithms should result in modal parameters identical to each other as well as the analytical solution. However, due to data processing steps involving windowing, averaging and singular value decomposition (SVD), the estimates differ from the analytical solution and also from each other. The same observation holds true for comparing the modal vectors.

From the formulations of the UMPA-SSI and UMPA-PTD, it is clear that the only difference between the two algorithms is that UMPA-SSI estimates the companion matrix directly, whereas UMPA-PTD estimates matrix coefficients which need to be assembled into the companion matrix for finding eigenvalues and eigenvectors. Bearing this in mind, the expectation is that the UMPA-SSI and UMPA-PTD should yield similar modal parameters - a fact also observable by means of this simple numerical example.

The difference in the modal parameter estimates of the two variants of the SSI algorithm (classical and UMPA-based) can be explained as follows. The SSI-COV contains an extra signal processing step involving SVD of the Hankel matrix and utilizes the observability matrix to obtain the modal parameters. However, this step is not a part of UMPA based formulation of the SSI algorithm (UMPA-SSI). Theoretically the two algorithms are still the same, although this extra signal processing step results in a slight difference in the estimates of the modal parameters.

It can be observed that cross MAC plots (as in Figures 2 to 4) comparing analytical, UMPA-SSI and SSI-COV vectors, are identical. A value of one in the diagonal means that modal vectors estimated using various algorithms are not only consistent (Allemang, 2003) in comparison to each other but also consistent with the analytical modal vectors. However, it should be noted (see Table 2) that the estimated modal vectors are not completely identical to the analytical modal vectors, though they are quite close to the theoretical values.

5. Conclusion

There are several modal parameter estimation algorithms and it is often difficult to understand the relationship between them. Although each algorithm has its own characteristic, the expectation is that they share a common thread; after all, they identify the same dynamic system. In this regard, the UMPA has been an attempt to demonstrate and develop various parameter estimation algorithms using a common mathematical framework. This work shows how the SSI algorithm can be derived within the UMPA framework. In its classical form, SSI algorithm seeks to identify the state transition matrix by performing singular value decomposition of the Hankel matrix of correlation functions and estimating the observability matrix. In this work it is shown that the state transition matrix

can also be identified directly from the Hankel matrix without estimating the observability matrix. The relationship of this state transition to the one obtained using the classical approach can be proved by means of a similarity transformation. This implies that both will have the same eigenvalues and eigenvectors and hence represent the same system.

Based on this understanding it is then shown how the basic time domain UMPA equation for OMA can be utilized to derive the SSI algorithm. It transpires that the SSI algorithm derived using the UMPA utilizes the alternative approach of obtaining the state transition matrix - i.e. its identification directly from the Hankel matrix. Furthermore, this formulation of the SSI algorithm is similar to a typical high order time domain OMA algorithm like the PTD algorithm; the only difference being that, while in the case of the PTD algorithm one estimates the polynomial coefficients which are then assembled in the form of a companion matrix to obtain modal parameters, UMPA-based formulation of the SSI algorithm results directly in the companion matrix itself. In other words, the state transition matrix obtained as a result of working on the Hankel matrix is merely the companion matrix that one will form once the polynomial coefficients are identified using PTD.

These results not only show the strength of the UMPA as a powerful concept for developing a variety of algorithms including the SSI algorithm, but also its usefulness in explaining the similarities between algorithms that otherwise seem very different from each other.

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