Subspace Algorithms in Modal Parameter Estimation for Operational Modal Analysis: Perspectives and Practices

S. Chauhan

Bruel & Kjær Sound and Vibration Measurement A/S

Skodsborgvej 307, DK 2850, Naerum, Denmark

Email: schauhan@bksv.com

Nomenclature

$\mathbf{x}$	State vector
$\mathbf{y}$	Response vector
$\mathbf{A}$	State Transition matrix
$\mathbf{C}$	Output matrix
$\mathbf{w}$ and $\mathbf{v}$	Process and measurement noise vectors
$\mathbf{Y}$	Hankel data matrix
$\mathbf{H}$	Hankel matrix of covariance matrices
$\mathbf{\Lambda}$	Covariance matrix
$\overline{\mathbf{O}}$	Extended Observability matrix
$\overline{\mathbf{C}}$	Extended Controllability matrix
$\mathbf{\Sigma}$	State Covariance matrix
$\overline{\boldsymbol{\alpha}}$	Matrix polynomial coefficients
$\mathbf{P}_{fp}$	Projection of future output responses on past output responses
$\hat{\mathbf{X}}_i$	Kalman filter state estimate

Abbreviations

OMA	Operational Modal Analysis
SSI	Stochastic Subspace Identification

SSI-Cov	Covariance-driven SSI
SSI-Data	Data-driven SSI
ITD	Ibrahim Time Domain
ERA	Eigensystem Realization Algorithm
SVD	Singular Value Decomposition

ABSTRACT

Subspace based algorithms for estimating modal parameter have now become common within modal analysis domain. This is especially true for Operational Modal Analysis, where Stochastic Subspace Identification (SSI) algorithm is a well-known and commonly used algorithm. Despite their increasing use and popularity, one often encounters basic questions such as (and not limited to)

1. How are these algorithms related to (or different from) traditional matrix polynomial coefficient based algorithms like Polyreference Time Domain (PTD) etc.?
2. What is the link between covariance and data driven approaches to SSI?
3. What is the need for having different variants of SSI (Covariance-driven and Data-driven)?

In fact, even before addressing the questions listed above, there is a fundamental need to look at these algorithms from the perspective of modal parameter estimation, whose requirements and demands differ from those of system identification within Control Systems Engineering, where these algorithms originated.

This paper aims at addressing these issues and examine subspace algorithms from a purely modal parameter estimation perspective. The author expects that this paper will provide readers with a simple and clear understanding of these algorithms towards their utilization for modal parameter estimation.

Keywords: Stochastic Subspace Identification, Data-driven, Covariance-driven, state space modelling, modal parameter estimation

1. Introduction

There are several applications, including modal parameter estimation where parametric models are sought. These include signal processing, defect detection, design of control systems and many others. Subspace algorithms belong to the class of system identification algorithms that utilize state-space models of time series for estimating parametric models. The use of these algorithms for parameter estimation in modal analysis is not new. Several popular algorithms, such as *Ibrahim Time Domain* (ITD) [1, 2] and *Eigensystem Realization Algorithm* (ERA) [3, 4], use state space formulations for estimating modal parameters.

The aim of this paper is to understand state space modelling in context of modal parameter estimation, with a focus on one of the most commonly used Operational Modal Analysis (OMA) algorithm, Stochastic Subspace Identification (SSI) [5-7], along with its two variants: Covariance-driven SSI (SSI-Cov) and Data-driven SSI (SSI-Data).

In order to understand state-space models in the context of modal analysis, it is vital to state the objective of modal analysis, which is ***to estimate modal parameters, i.e. natural frequency, damping and mode shape, of a structure***. Thus, unlike system identification problem in control system design or forecasting problem in econometrics, the goal of a parameter estimation algorithm is not estimation of matrices **A**, **C** (defined in section 2) or associated quantities such as state vectors, observability or controllability matrix, etc. but modal parameters.

The paper starts with a general discussion on SSI algorithm and describes sequentially its two variants, Covariance-driven SSI (SSI-Cov) and Data-driven SSI (SSI-Data). In section 2.1, two separate approaches are provided for the development of SSI-Cov. First, traditional formulation of SSI-Cov is provided along with a modified formulation. Then an alternate formulation is provided that takes inspiration from, and explores, the relationship between the polynomial model and state-space model representation of a dynamic system. Additionally, this section explains how estimation of extended observability matrix in the conventional formulation is not a requirement from modal analysis point of view and desired results can be achieved in comparatively simple steps. The section concludes by noting how various formulations of SSI-Cov simply differ in how the covariance matrices are stacked.

Same approach is taken while discussing SSI-Data (Section 2.2). Given the framework of modal parameter estimation, the aim is to connect the two variants of SSI. In this context, it is explained how the need to estimate state vectors necessitates the availability of raw output data for SSI-Data algorithm and it is emphasized that these requirements do not form a part of modal parameter estimation. The formulation of SSI-Data is then explained in light of this knowledge and it is shown how SSI-Data is not much different from SSI-Cov in the absence of the need to estimate state vectors.

2. Stochastic Subspace Identification algorithm

Stochastic Subspace Identification (SSI) is a well-known operational modal analysis (OMA) algorithm [5-7]. It is based on state-space representation of a discrete linear time invariant (LTI) system described as

$$\begin{aligned} \mathbf{x_{k+1}} &= \mathbf{A}\mathbf{x_k} + \mathbf{w_k} \\ \mathbf{y_k} &= \mathbf{C}\mathbf{x_k} + \mathbf{v_k} \end{aligned} \tag{01}$$

where $\mathbf{y}$ is the vector of measured responses, $\mathbf{x}$ is the vector of state variables matrix, $\mathbf{A}$ is the state transition matrix, $\mathbf{C}$ is the output matrix and $\mathbf{w}$ and $\mathbf{v}$ are process and measurement noise vectors. As is clear from Eqn. 01, SSI algorithm operates on measured output response data only (input excitation is not measured in OMA). From the point of view of modal parameter estimation, it is the estimation of state transition matrix $\mathbf{A}$ that is most important as eigenvalue decomposition of this matrix reveals modal parameters.

There are two variants of this algorithm that are popular in practice: data-driven (SSI-Data) and covariance-driven (SSI-Cov) [6]. These variants differ in terms of the data on which they operate. SSI-Data operates directly on measured output response data without processing it. On the other hand, SSI-Cov requires that covariance functions are first estimated from raw output time histories and it is these covariance functions that SSI-Cov utilizes for the purpose of modal parameter estimation. The discussion in this section centers on understanding the finer aspects of these two variants. In this context, traditional development of these algorithms is studied to understand the reasons that led to the development of these two variants.

Typically, the output response data is assembled in a Hankel data matrix, which is further divided into a partition of past ($\mathbf{Y}_p$) and future responses ($\mathbf{Y}_f$) (see [6, 7] for more details). This arrangement aids in theoretical formulation of SSI and is given as

$$\mathbf{Y} = \begin{bmatrix} \mathbf{Y}_p \\ \mathbf{Y}_f \end{bmatrix} \tag{02}$$

The two variants of SSI are now discussed in section 2.1 and 2.2.

2.1 Covariance driven Stochastic Subspace Identification algorithm

SSI-Cov algorithm, as the name suggests, operates on covariance functions. Typically, the raw response data is processed into covariance functions, which are then utilized in SSI-Cov algorithm. This processing of raw response data to covariance (or correlation) functions can be done in several ways, including obtaining covariance functions directly from response data without involving any intermediary signal processing steps. This is in contrast to, for e.g. Welch Peridogram method [8], where data is processed to estimate power spectra, which is then inverse Fourier transformed to obtain covariance functions, and involves signal-processing techniques like averaging, windowing etc.

In literature, SSI-Cov is generally developed using the Hankel (or Toeplitz) matrix of covariance functions. The algorithm then utilizes the fact that the Hankel matrix equals the covariance between future responses and past responses, i.e. $\mathbf{H} = \mathbf{Y}_f \mathbf{Y}_p^T$. In case of SSI-Cov, it is typically assumed that covariance functions are calculated a priori and access to raw output responses is not available.

This section is arranged in the following manner. First, the traditional formulation of SSI-Cov, based on estimation of extended observability and controllability matrices, is presented. It is then shown that estimation of extended observability and controllability matrices is redundant from the perspective of modal parameter estimation and how state transition matrix $\mathbf{A}$ can be directly obtained from Hankel matrix of covariance functions. Finally, a new formulation of SSI-Cov is suggested in section

2.1.2 underlining the fact that SSI-Cov can be formulated in multiple ways; each differing in the manner the covariance functions are arranged.

2.1.1 SSI-Cov based on traditional formulation

Traditional formulation of SSI-Cov begins with formulation of a Hankel (or Toeplitz) matrix of covariance matrices, which equals $\mathbf{H} = \mathbf{Y}_f \mathbf{Y}_p^T$ ($\mathbf{Y}_f, \mathbf{Y}_p$ are defined in Eqn. 02).

$$\mathbf{H} = \mathbf{Y}_f {\mathbf{Y}_p}^T = \begin{bmatrix} \mathbf{\Lambda}_1 & \mathbf{\Lambda}_2 & \cdots & \mathbf{\Lambda}_m \\ \mathbf{\Lambda}_2 & \mathbf{\Lambda}_3 & \cdots & \mathbf{\Lambda}_{m+1} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{\Lambda}_m & \mathbf{\Lambda}_{m+1} & \cdots & \mathbf{\Lambda}_{2m} \end{bmatrix}_{mN_o \times mN_o} \tag{03}$$

As explained previously, for modal parameter estimation purposes, one is interested in estimation of state transition matrix **A**. Typically, estimation of state transition matrix **A** is done on the basis of extended Observability matrix $\overline{\mathbf{O}}$, which is obtained using singular value decomposition (SVD) of **H**.

$$\mathbf{H} = \mathbf{U}\,\mathbf{S}\,\mathbf{V}^T = \overline{\mathbf{O}}\,\overline{\mathbf{C}} \tag{04}$$

Note that $\overline{\mathbf{C}}$ in the Eqn. 04 represents extended Controllability matrix. The extended observability matrix is obtained in following manner,

$$\overline{\mathbf{O}} = \mathbf{U}\,\mathbf{S}^{½} \tag{05}$$

and state transition matrix **A** is estimated as

$$\mathbf{A} = \vec{\mathbf{O}}\,\overline{\mathbf{O}}\,^{+}_{\text{m-1}} \tag{06}$$

where $\vec{\mathbf{O}}$ is the block shifted version of extended observability matrix (obtained after removing the first block of $\overline{\mathbf{O}}$) and $\overline{\mathbf{O}}\,^{+}_{\text{m-1}}$ represent pseudo-inverse of first (m-1) rows of $\overline{\mathbf{O}}$.

The reason for taking this approach in conventional system identification framework is that identification of observability and controllability matrices is often part of the overall identification problem. This however, is not the case with modal parameter estimation, where the aim is to obtain modal parameters. Shown below is another approach of estimating state transition matrix **A**, using the Hankel matrix expressed in Eqn. 03. This formulation is based on the work presented in [9], where it is shown that state transition matrix **A**, can be directly obtained from Hankel matrix **H** in following manner.

$$\mathbf{A} = \overrightarrow{\mathbf{H}}\,\mathbf{H}\,^{+}_{\text{m-1}} \tag{07}$$

where $\overrightarrow{\mathbf{H}}$ is the block shifted version of Hankel matrix and $\mathbf{H}\,^{+}_{\text{m-1}}$ represent pseudo-inverse of first (m-1) rows of **H**.

It should be noted that state transition matrix estimated using Eqn. 07 is related to that obtained using the traditional approach (described by Eqn. 04-06) by means of a similarity transformation (see [9] for details) and hence possesses the same Eigen structure. In other words, modal parameters estimated on the basis of either state transition matrix will still be the same.

2.1.2 Alternate formulation

By defining state vectors $\mathbf{x_k}$, in Eqn. 01, in following manner, it is easy to understand the relationship between state-space representation and the polynomial model representation of a dynamic system. The state transition matrix in this case becomes a companion matrix.

$$\mathbf{x}_k = \begin{bmatrix} \mathbf{y}_{k+1} \\ \mathbf{y}_{k+2} \\ \vdots \\ \mathbf{y}_{k+m} \end{bmatrix}, \qquad \mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -\overline{\boldsymbol{\alpha}}_0 & -\overline{\boldsymbol{\alpha}}_1 & -\overline{\boldsymbol{\alpha}}_2 & \cdots & -\overline{\boldsymbol{\alpha}}_{m-1} \end{bmatrix} \tag{08}$$

Defining covariance matrix between output responses as $\boldsymbol{\Lambda}_i = E[\mathbf{y}_{k+i}\mathbf{y}_k^T]$ and state covariance matrix as $\boldsymbol{\Sigma} = E[\mathbf{x}_k\mathbf{x}_k^T]$, it is easy to show, using Eqn. 01, that

$$\boldsymbol{\Lambda}_i = \mathbf{C}\mathbf{A}^{i-1}\boldsymbol{\Sigma}\mathbf{C}^T \tag{09}$$

Expanding state equation in Eqn. 01 and using Eqn. 08,

$$\begin{bmatrix} \mathbf{0} & \mathbf{I} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{I} \\ -\boldsymbol{\alpha}_0 & -\boldsymbol{\alpha}_1 & -\boldsymbol{\alpha}_2 & \cdots & -\boldsymbol{\alpha}_{\mathbf{m-1}} \end{bmatrix} \begin{bmatrix} \mathbf{y}_{i+0} \\ \mathbf{y}_{i+1} \\ \vdots \\ \mathbf{y}_{i+m-2} \\ \mathbf{y}_{i+m-1} \end{bmatrix} = \begin{bmatrix} \mathbf{y}_{i+1} \\ \mathbf{y}_{i+2} \\ \vdots \\ \mathbf{y}_{i+m-1} \\ \mathbf{y}_{i+m} \end{bmatrix} \tag{10}$$

Since the state transition matrix **A** is constant (the system is time invariant), a number of equations similar to Eqn. 10 can be formed as shown below.

$$\begin{bmatrix} \mathbf{0} & \mathbf{I} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ -\boldsymbol{\alpha}_0 & -\boldsymbol{\alpha}_1 & \cdots & -\boldsymbol{\alpha}_{\mathbf{m-1}} \end{bmatrix} \begin{bmatrix} \mathbf{y}_{i+0} & \mathbf{y}_{i+1} & \cdots & \mathbf{y}_{i+m-1} \\ \mathbf{y}_{i+1} & \mathbf{y}_{i+2} & \cdots & \mathbf{y}_{i+m} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{y}_{i+m-1} & \mathbf{y}_{i+m} & \cdots & \mathbf{y}_{i+2(m-1)} \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{y}_{i+1} & \mathbf{y}_{i+2} & \cdots & \mathbf{y}_{i+m} \\ \mathbf{y}_{i+2} & \mathbf{y}_{i+3} & \cdots & \mathbf{y}_{i+m+1} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{y}_{i+m} & \mathbf{y}_{i+m+1} & \cdots & \mathbf{y}_{i+2m-1} \end{bmatrix} \tag{11}$$

For the sake of simplicity, Eqn. 11 can be written in a compact form as

$$\mathbf{AP} = \mathbf{F} \tag{12}$$

which can be solved for **A** in a least squares manner as,

$$\mathbf{A} = \mathbf{FP}^T\left[\mathbf{PP}^T\right]^{-1} \tag{13}$$

The two products in the above equation, $\mathbf{FP}^T, \mathbf{PP}^T$ are two Toeplitz matrices comprising covariance functions

$$\mathbf{PP}^T = \begin{bmatrix} \Lambda_0 & \Lambda_1 & \cdots & \Lambda_{m-1} \\ \Lambda_{-1} & \Lambda_0 & \cdots & \Lambda_{m-2} \\ \vdots & \vdots & \ddots & \vdots \\ \Lambda_{-(m-1)} & \Lambda_{-(m-2)} & \cdots & \Lambda_0 \end{bmatrix}, \qquad \mathbf{FP}^T = \begin{bmatrix} \Lambda_1 & \Lambda_2 & \cdots & \Lambda_m \\ \Lambda_0 & \Lambda_1 & \cdots & \Lambda_{m-1} \\ \vdots & \vdots & \ddots & \vdots \\ \Lambda_{-m} & \Lambda_{-(m-1)} & \cdots & \Lambda_1 \end{bmatrix}$$

It is recognizable that the two matrices $\mathbf{FP}^T, \mathbf{PP}^T$ have similar structure as $\vec{\mathbf{H}}, \mathbf{H}_{m-1}$, described in Eqn. 07. This explains the connection between the traditional approach (based on $\mathbf{H} = \mathbf{Y}_f\mathbf{Y}_p^T$) and the approach suggested in this paper.

Based on above observations, it can be argued that there is no fundamental difference between various formulations of SSI-Cov. The biggest difference is perhaps how the covariance matrices are formed and stacked, and as shown in this paper, this can be done in several ways without having any impact on the final outcome in terms of estimation of modal parameters.

2.2 Data driven Stochastic Subspace Identification algorithm

Formulation of SSI-Cov algorithm assumes that covariance functions are available and raw output data does not play any role irrespective of whether it is available or not. Various formulations shown in previous section assume that raw data is available, but it is easily noticeable that this is not a requirement and, starting with Eqn. 03 (or Eqn. 13) the formulations can be arrived at without much difficulty by simply using the covariance functions in case they are available a priori. SSI-Data, on the contrary, makes it mandatory to have the raw data available. This is the oft-quoted difference between the two popular variants of SSI. However, it can be argued that this requirement is driven more by the need to estimate state vectors $\mathbf{x}$ within controls engineering domain than by requirements associated with modal parameter estimation. Estimation of state vectors is typically done by means of Kalman filter [5-7], which provides optimal prediction of state vector. It is this requirement that necessitates the availability of raw output time data.

SSI-Data is based on the concept of projection [5, 10], where future outputs are projected on the past outputs. This projection is defined as

$$\mathbf{P}_{fp} = \left[\mathbf{Y}_f \mathbf{Y}_p^T\right]\left[\mathbf{Y}_p \mathbf{Y}_p^T\right]^{+} \mathbf{Y}_p \tag{14}$$

The Kalman filter state estimates are given as

$$\hat{\mathbf{X}}_k = \overline{\mathbf{C}} \left[\mathbf{Y}_p \mathbf{Y}_p^T\right]^{+} \mathbf{Y}_p \tag{15}$$

Using Eqns. 03-04, it is easy to follow that projection defined in Eqn. 14 can be written as

$$\begin{aligned} \mathbf{P}_{fp} &= \overline{\mathbf{O}}\,\overline{\mathbf{C}} \left[\mathbf{Y}_p \mathbf{Y}_p^T\right]^{+} \mathbf{Y}_p \\ &= \overline{\mathbf{O}}\,\hat{\mathbf{X}}_i \end{aligned} \tag{16}$$

This paves the way for decomposing projection $\mathbf{P}_{fp}$ such that extended observability matrix $\overline{\mathbf{O}}$ and Kalman filter states $\hat{\mathbf{X}}_i$ can be estimated. This is done by means of singular value decomposition of $\mathbf{P}_{fp}$. It is noticeable that estimation of extended observability matrix remains same.

$$\begin{aligned} \overline{\mathbf{O}} &= \mathbf{U}\,\mathbf{S}^{1/2} \\ \hat{\mathbf{X}}_i &= \mathbf{S}^{1/2}\mathbf{V}^T \end{aligned} \tag{17}$$

The state transition matrix can now be easily obtained using the expression provided in Eqn. 06. However, it is common to provide an alternate expression based on the estimated Kalman filter state and its shifted version $\hat{\mathbf{X}}_{k+1}$. Using Eqn. 01 it is easy to obtain **A**, in terms of estimated state vectors, as

$$\hat{\mathbf{X}}_{k+1}\,\mathbf{A} = \hat{\mathbf{X}}_k \tag{18}$$

It is clear from above discussion that the reason for using SSI-Data is estimation of state vectors. However, as has been mentioned, this is not a requirement from modal parameter estimation perspective. More importantly, even after estimation of Kalman filter states, one can still utilize the extended observability matrix for estimating state transition matrix [7]. Thus, if estimating the state vectors is not the goal, it is easy to follow that SSI-Data can be formulated simply on the basis of extended observability matrix. In that case, there is no difference between this approach and the one expressed in previous section that described SSI-Cov. Note that basic formulation of SSI-Cov is based on covariance functions and one cannot estimate state vectors only on the basis of covariance functions.

The major distinction between the two variants of SSI is in terms of implementation; unlike SSI-Cov, SSI-Data can be implemented such that calculation of covariance matrices is not required [6]. Otherwise, the only distinction between the two approaches is that in case of SSI-Cov, covariance functions are supposed to be pre-calculated, which is not the case with SSI-Data. SSI-Cov starts directly from the formation of Hankel matrix **H** (Eqn. 03) where as in SSI-Data the raw output data is arranged in terms of past and future responses (Eqn. 02) so that the covariance functions are calculated as a part of the algorithm.

3. Conclusions

This paper reviews the Stochastic Subspace Identification (SSI) algorithm and its two variants (SSI-Cov and SSI-Data) within the framework of modal parameter estimation. One underlining aspect of SSI pointed out in this paper is the fact that the goals of modal parameter estimation stage of operational modal analysis, i.e. estimation of natural frequency, damping and unscaled mode shape, can be achieved through several formulations of SSI. This fact is highlighted by means of an alternate formulation of SSI-Cov presented in this paper. This is done by exploring the relationship between state-space model and high order polynomial model representation of a dynamic system.

The paper further emphasizes that, when viewed within the framework of modal parameter estimation, there is not much difference between SSI-Cov and SSI-Data. This is due to the fact that the aim of these variants differ from when they are applied in Controls Engineering domain (where these algorithms were originally developed) to their application in modal analysis domain. It is understandable from the formulation of SSI-Data that one of its primary goal is state estimation, a goal that is not shared by modal parameter estimation. That state transition matrix can be obtained using the estimated states is an auxiliary step, as estimation of state transition matrix does not rely solely on state vector. State transition matrix can be obtained simply on the basis of extended observability matrix, in which case, there is not much distinction between SSI-Data and SSI-Cov.

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